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ABSTRACT

This student textbook is designed to incorporate programable calculators in grade 11 mathematics. The eight chapters contained in this document are: (1) Using Calculators in Mathematics; (2) Exponents and Logarithms; (3) Trigonometry of the Right Triangle; (4) Trigonometry Beyond the Right Triangle; (5) Graphs of the Trigonometric Functions and Their Inverses; (6) Solution of Oblique Triangles and Other Applications of Trigonometry; (7) The Quadratic and Other Polynomial Functions; and (8) Sequences and Series. Each chapter is further subdivided into topic sections. Each section concludes with a set of calculator-oriented exercises geared towards the material covered. (MP)

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SE 035 940

USING CALCULATORS IN MATHEMATICS

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CHAPTER 1. USING CALCULATORS IN MATHEMATICS

In this chapter you will become familiar with different kinds of calculators and you will learn how to compute with algebraic, AOS, RPN, and arithmetic calculator logics. You will also learn simple programming.

1.1 Order of Operations

Communication of ideas is important in mathematics. The reader of mathematics must understand what the writer of mathematics means. For this reason we adopt rules for writing and reading that are generally accepted. For example, when we write

$$5 + 2 \times 3$$

we want all readers to interpret what we have written in the same way. Of the two choices

(a)	(b)
$5 + 2 \times 3$	$5 + 2 \times 3$
7×3	$5 + 6$
21	11

you would probably choose (b), answer 11, because you recall rules for order of operations. Unfortunately a younger brother or sister in elementary school or an adult who has not studied school mathematics (or has forgotten it) would probably choose (a). Thus rules designed to improve communication sometimes fail. We will review those rules and see how calculators force us once again to watch our step.

ORDER OF OPERATIONS RULE. Apply operations in the following order:

(1) within parentheses
(2) exponentiation (powers and roots)
(3) multiplication and division*
(4) addition and subtraction

---

*In order to avoid rare instances where confusion might arise, some authors insist upon multiplication before division. We do not adopt that convention in this text.

Students sometimes remember this rule by the mnemonic:

Please Enter, My Dear Aunt Sally.

Only in the absence of rule priorities do you calculate left to right.

EXAMPLE 1. Evaluate $2 \cdot 4^2 - 6 \div \frac{\sqrt{9}}{7} + 5 \cdot 8$

Solution: $2 \cdot 4^2 - 6 \div \frac{\sqrt{9}}{7} + 5 \cdot 8$

$2 \cdot 16 - 6 \div \frac{3}{7} + 5 \cdot 8$ exponentiation

$32 - 14 + 40$ multiplication and division

58 addition and subtraction

EXAMPLE 2. Evaluate $\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2}$

Solution: $\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2}$

$\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 9}$ exponentiation

$\frac{10 + 12}{20 - 9}$ multiplication. Note that the fraction bar (vinculum) plays a role as parentheses.* Thus numerator and denominator are simplified before division.

$\frac{22}{11}$ addition and subtraction

2 division

The examples have been worked out in detail. In practice many of these steps would be skipped. For example the second solution might be recorded as

---

*Another example of this usage is in roots like $\sqrt{x + y}$. The root symbol is $\surd$ and the bar is a grouping symbol. In Europe $\sqrt{x + y}$ is often written $\surd(x + y)$.

$$\frac{2 \cdot 5 + 3 \cdot 4}{10 \cdot 2 - 3^2} = \frac{22}{11} = 2$$

Exercise Set 1.1

1 - 8 Evaluate:

(1) $2 \cdot 3 + 6$ (2) $2(3 + 6)$

(3) $6 + 2 \cdot 3$ (4) $(6 + 2)3$

(5) $7 + 2 \cdot 5$ (6) $(7 + 2)5$

(7) $2 \cdot 5 + 7$ (8) $2(5 + 7)$

9 - 16 Some writers use parentheses as "insurance" to guarantee that readers will calculate in the desired order. When it is possible in each of the following, write an equivalent expression without parentheses:

(9) $(ab) + (cd)$ (10) $(a + b)(c + d)$

(11) $(\frac{a}{b} \div \frac{c}{d}) \times \frac{e}{f}$ (12) $\frac{a}{b} \div (\frac{c}{d} \times \frac{e}{f})$

(13) $a\left[b + c\ (d + e)\right]$ (14) $\left\{\left[(a + b) + c\right] - d\right\}$

(15) $\left\{\left[(ab) + c\right] d\right\} + e$ (16) $\frac{\sqrt{(a + b)}}{\left[(cd) - e\right]}$

17. For each of the exercises 9 - 16, evaluate (a) the original expression and (b) your simplified expression for the values $a = 6$, $b = 3$, $c = 4$, $d = 2$, $e = 7$, $f = -1$.

18 - 24. Notice in the following exercises how order makes no difference in exercises involving addition and subtraction, but seems to in exercises involving multiplication and division. Evaluate:

(18) $2 - 3 + 5$ (19) $2 + 5 - 3$

(20) $-3 + (5 + 2)$ (21) $2 \div 10 \times 5$ Be careful!

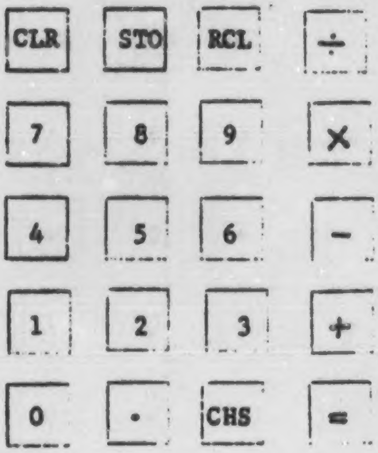
(22) $2 \times 5 \div 10$ (23) $2 \div (10 \times 5)$

1.2 Calculator Logic: Algebraic with Memory

The calculator user must learn how to process numbers on the specific instrument he is using. This is important because of differences among calculators. In this section and the next three, we introduce several common calculator "logics". Most calculators operate by one of them or by a minor variation. A calculator user tends to become accustomed to the logic of his machine and to prefer it. Indeed, each has certain advantages which we will consider. Even if you will be using a calculator with a particular logic system, it is important to know how the others work. Who knows what kind you'll be using next?

ALGEBRAIC LOGIC

Algebraic logic is a common calculator logic. The figure displays a keyboard for a simple algebraic logic calculator.* (Do not look for a calculator with this exact display as most have additional keys like $\boxed{\%}$ and $\boxed{\sqrt{}}$ that are useful but not necessary to this discussion.) Some keys are marked with common abbreviations: CLR - clear; STO - store, RCL - recall, CHS - change sign. This last key is sometimes marked $\boxed{+/-}$ instead.



The logic of this machine is called algebraic but it doesn't follow the order rules of algebra you learned in Section 1.1. Calculations are fed into the machine much as you would type them on a typewriter (without spacing). Thus the multiplication 23 × 56 would be keyed

$$\boxed{2}\ \boxed{3}\ \boxed{\times}\ \boxed{5}\ \boxed{6}\ \boxed{=}$$

An instant after the $\boxed{=}$ is pressed the calculator displays the product

1288*

Chains of operations may also be keyed directly into the calculator under certain conditions.

EXAMPLE 1. Calculate $21 \times 32 \times 61 \div 24$

Solution: Key:

$\boxed{2}$ $\boxed{1}$ $\boxed{\times}$ $\boxed{3}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{6}$ $\boxed{1}$ $\boxed{\div}$ $\boxed{2}$ $\boxed{4}$ [illegible]

Final Display: 1708.

EXAMPLE 2. Calculate 3.1 - 5.7 + 4.6

Solution: Key

$\boxed{3}$ $\boxed{\cdot}$ $\boxed{1}$ $\boxed{-}$ $\boxed{5}$ $\boxed{\cdot}$ $\boxed{7}$ $\boxed{+}$ $\boxed{4}$ $\boxed{\cdot}$ $\boxed{6}$ $\boxed{=}$

Display: 2.0 (or 2)

It is both interesting and useful to note that intermediate results are displayed on the calculator at various points in these chains of operations. In Example 1, for instance, when the second $\boxed{\times}$ is keyed in the calculation the display becomes

672

which is the product 21 × 32, the first two factors. Similarly when the $\boxed{\div}$ is keyed the display changes to

40992

---

*We will not attempt to replicate calculator displays in this text. Machines differ widely; most, however display numbers by lighting or filling with liquid some or all of seven small bars 8. These may be seen by looking closely. You might like to determine how many different displays could be made with these seven bars.

the result of the calculation to this point ($21 \times 32 \times 61 = 40992$). In exactly the same way in Example 2 the intermediate result

$$-2.6$$

is displayed when the $\boxed{+}$ is keyed.

It should be clear that algebraic logic is fine for chained computations that process left to right. But we saw in Section 1.1 that many computations do not have this simple order. Such computations lead to problems. To detect these problems the user must be alert; to solve them ingenuity must be exercised. The user must supply the one thing the calculator cannot: thinking! (This last sentence will, in fact, be a central message in all that follows.)

Consider the calculation

EXAMPLE 3. $$\frac{2 \times 3}{5 \times 4}$$

We know that the answer to this calculation is 3/10 or .3, and we would expect the calculator to display 0.3. You might attempt to carry this out by the following sequence.

$$\boxed{2}\quad\boxed{\times}\quad\boxed{3}\quad\boxed{\div}\quad\boxed{5}\quad\boxed{\times}\quad\boxed{4}\quad\boxed{=}$$

The result of this sequence is 4.8, the wrong answer. Can you see what is incorrect in the calculation? The error is identified if the fraction is represented differently:

$$\frac{2 \times 3}{5 \times 4} = 2 \times 3 \times \frac{1}{5 \times 4} = 2 \times 3 \times \frac{1}{5} \times \frac{1}{4} = 2 \times 3 \div 5 \div 4$$

Thus, in general, each factor of the denominator is a divisor. This is a useful calculating technique to remember.

A corrected calculation is

$\boxed{2}\quad\boxed{\times}\quad\boxed{3}\quad\boxed{\div}\quad\boxed{5}\quad\boxed{\div}\quad\boxed{4}\quad\boxed{=}$

giving the correct result

0.3

A more difficult problem is presented by a calculation like:

EXAMPLE 4. $\dfrac{49 + 38}{85 + 96}$

This time we have no direct solution technique. Several alternatives are available:

Solution 4-1. Calculate 49 + 38. Record the answer 87 on a scratch pad.
Calculate 85 + 96. Record this answer 181.
Calculate 87 ÷ 181. This quotient, 0.4807, *
is the answer to the exercise.

There is nothing wrong with the solution shown here, but such a solution does not use the full power of the calculator. It is more than a matter of elegance not to have to write down such intermediate answers. Time may be lost and additional opportunities for error are accumulated as you copy and reenter numbers. Use of calculator storage (or memory) provides an alternative.

Solution 4-2.

$\boxed{8}\quad\boxed{5}\quad\boxed{+}\quad\boxed{9}\quad\boxed{6}\quad\boxed{=}$ Calculate the denominator, 181.

$\boxed{\text{STO}}$ Store this number in calculator memory.

$\boxed{4}\quad\boxed{9}\quad\boxed{+}\quad\boxed{3}\quad\boxed{8}\quad\boxed{=}$ Calculate the numerator, 87.

---

*Results in this text will be given for 4-digit rounding displays.

$\boxed{\div}$ $\boxed{\text{RCL}}$ $\boxed{=}$ $\boxed{\text{RCL}}$ brings back the number, 181, from memory.

Exercise Set 1.2

Some of these exercises call for a calculator with simple algebraic logic with memory. (If your calculator has parenthesis keys, do not use them.)

1. Name four other keys that could replace $\boxed{=}$ in the calculation of Examples 1 and 2 to give the same answer. State a reason why you would not use these substitute keys if you were carrying out a series of calculations. (Try calculating 2 × 3 followed by 3 + 4.)

2. One step in Solution 4-2 of the text may be eliminated. Examine the calculation carefully in order to find the extra step. Check your more elegant solution on a calculator.

3. In Solution 4-2 we calculated the denominator first. Try calculating the numerator first. What happens? (Some more sophisticated calculators have a key that switches the contents of two registers to avoid this kind of trap.)

4-10 Calculate, keeping intermediate record keeping to a minimum. Note which exercises require such records. RECALL THE ORDER RULES FROM SECTION 1.1.

4. $237 \times 42.5 + 38.46$

5. $39.42 + 861.7 \times 6.03$

6. $23.7 \div .06 \times 13.2$

7. $(78.35 + 91.46)(14.08 - 27.61)$

8. 2.8^3 Try to find an elegant way to calculate this.

9. $\dfrac{37.48 - 16.89}{64.32}$

10. $\dfrac{64.32}{37.48 - 16.89}$

11. How should the answers to exercises 9 and 10 be related? Check this by calculation.

12-16 Calculate. Note intermediate records.

12. $$\frac{239.5 - 67.34}{(74.2)(86.3)}$$

13. $$\frac{(74.2)(86.3)}{239.5 - 67.34}$$

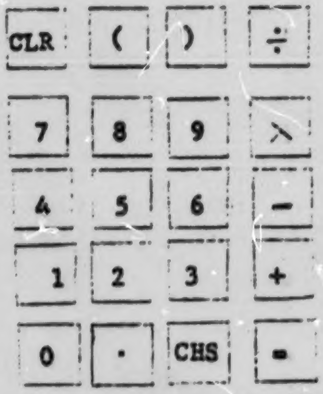
14. $(37.6 - 18.4)(15.2 - 83.1)(64.2 + 73.8)$ Beware: Some algebraic calculators allow the user only to add to or subtract from memory. If you are using one of those calculators, be sure to clear memory before storing a second number.

15. $$\frac{(37.6 - 18.4)(15.2 - 83.1)}{64.2 + 73.8}$$

16. $$\frac{4231(16.8 - 23.4)}{(83 - 1.3752)\,62.43}$$

1.3 Calculator Logic: Algebraic with Parentheses*

The simple addition of parentheses to the algebraic keyboard simplifies much computation. The figure displays a keyboard for a calculator operating with this logic which we will call (algebraic). Almost all such calculators have additional features like STO , RCL , √x , % and 1/x , but we restrict our discussion to the ones shown.



A quick comparison of this keyboard with the keyboard of Section 1.2 shows that only two keys are different: STO and RCL are replaced by (and). Surprisingly this minor modification makes keying of complex calculations much simpler.

The main point to remember: Parentheses on the calculator play the same role of grouping computations that they do in algebra. There is, however, a difference in usage. The algebraic expression

$$a \{ b + (cd - e) \}$$

would appear as the calculator expression

$$a(b + (cd - e)).$$

Thus braces, brackets and other grouping symbols are all represented by the same symbols, parentheses.

---

*In sections 1.3 through 1.5 we will refer to algebraic logic with parentheses as (algebraic) in order to differentiate it from the algebraic memory logic of section 1.2.

EXAMPLE 1. $\dfrac{49 + 38}{85 + 96}$

(Recall that this was Example 4 of Section 1.2)

Solution:

$\boxed{4}\ \boxed{9}\ \boxed{+}\ \boxed{3}\ \boxed{8}$ Calculate the numerator

$\boxed{\div}$ Divided by... (Numerator displayed: 87)

$\boxed{(}$ ... the quantity...(signals a calculation to be done out of sequence.)

$\boxed{8}\ \boxed{5}\ \boxed{+}\ \boxed{9}\ \boxed{6}$ Calculate the denominator

$\boxed{)}$ Completes the calculation in parentheses and displays it (181)

$\boxed{=}$ Displays the quotient of $87 \div 181$, 0.4807

Notice the effect of the right parenthesis, $\boxed{)}$:

(1) It plays the role of the $\boxed{=}$ key for the calculation since the most recent left parenthesis $\boxed{(}$ and displays the result.

(2) It "backs up" the calculation to where the left parenthesis $\boxed{(}$ was keyed. Thus the calculator acts as though you had just entered the calculated value of what is in parentheses.

WARNING: Parentheses do NOT represent multiplication!

EXAMPLE 2. $-4.9\ (3.7 - 8.9)$

Solution:

$\boxed{4}\ \boxed{\cdot}\ \boxed{9}\ \boxed{\text{CHS}}$	Enter the multiplier, -4.9
$\boxed{\times}$	Multiplied by...(since the parentheses do not carry this meaning)
$\boxed{(}\ \boxed{3}\ \boxed{\cdot}\ \boxed{7}\ \boxed{-}$	Calculates the value of the
$\boxed{8}\ \boxed{\cdot}\ \boxed{9}\ \boxed{)}$	expression in parentheses (-5.2)
$\boxed{=}$	Displays the product of -4.9 and -5.2

25.48

A modified algebraic logic that is closer to the rules of section 1.1 is called AOS logic. The letters represent the words Algebraic Operating System. With AOS logic calculators the calculation

$$3 + 5 \times 7$$

could be keyed left to right without parentheses.

$$\boxed{3}\ \boxed{+}\ \boxed{5}\ \boxed{\times}\ \boxed{7}\ \boxed{=}$$

The calculator "remembers" when the $\boxed{\times}$ key is pressed that multiplication takes precedence over addition.

AOS calculators also require either memory or parentheses to process exercises like Example 1 of page 1.3 - 2. On an AOS calculator with parentheses the calculation would be exactly as given.

Exercise Set 1.3

Rewrite each of the following expressions, for (algebraic) logic:

(a) removing parentheses that will not change the value algebraically

(b) removing parentheses that will not change the value in calculator computation

(1) $3 + (5 - 7)$

(2) $20 \times (10 \div 5)$

(3) $\frac{(2 \times 7)}{(31 - 14)}$

(4) $20 \div (10 \times 5)$

(5) $(8 + 7)(3 + 5)$

(6) $(27.3 + 41.7)3.6$

(7) $27.3 + (41.7 \times 3.6)$

(8) $(41.7 \times 3.6) + 27.3$

(9) $41.7 \times (3.6 + 27.3)$

(10) $\frac{(28 \times 3) + 8}{((26 + 7) \times 4)}$

Compute with an (algebraic) calculator:

(11) $37.8 + (.06 \times 37.8)$

(12) 1.06×37.8

(13) $(2.8 \div 4.5)^2(16 \div 39.23)^2$

(14) $\frac{26.4}{.0631 - .1256}$

Calculate by algebraic-memory, by (algebraic) and by (AOS) to compare procedures

(15) $\frac{327.84}{264 - 189}$

(16) $(48.3 + 27.9)(79.4 - 43.7)(67.1 - 4)$

(17) In the song "The Twelve Days of Christmas", the lyrics begin:

On the first day of Christmas
My true love gave to me
A partridge in a pear tree

On the second day are given:

Two French hens and a partridge

On the third day:

Three turtle doves, two French hens and a partridge.

So it goes through twelve days until on the twelfth, for example, she receives:

> Twelve lords aleaping, eleven ladies waiting, ...
> [all the way down to] ... a partridge in a pear tree

Now suppose that on Christmas day the lovers break up and the gifts are returned one each day. For example, on Christmas day one of of the partridges might be returned, the next day another, the following day another, the following day a French hen, and so on. When will all the gifts be returned?

Set this problem up carefully and solve it by calculator.

1.4 Calculator Logic: RPN

The letters RPN represent Reverse Polish Notation, the country designated because the Polish logician J. Łukasiewicz developed the system. RPN is, in fact, often called Łukasiewicz logic. The reason for the R (Reverse) is that in this notation operation symbols are applied in order that is the reverse of what we learn in arithmetic and algebra. Thus

3 + 4 in RPN is 3 4 +

Think about that notation for a minute. What would happen if you keyed into any calculator:

$\boxed{3}$ $\boxed{4}$ $\boxed{+}$?

It would record the number 34. Because of this problem an additional key appears on RPN calculator keyboards, the ENTER (or ENT) key. Thus 3 + 4 is keyed:

$\boxed{3}$ $\boxed{\text{ENT}}$ $\boxed{4}$ $\boxed{+}$

On many keyboards the ENTER key is larger because it is used so often.

Why would anyone want to change things around like that? It turns out that there are good reasons. If you examine Algebraic and RPN keyboards, you will see that the RPN ENTER replaces three algebraic keys:

RPN	Algebraic or AOS
$\boxed{\text{ENT}}$	$\boxed{=}$ $\boxed{(}$ $\boxed{)}$

We will now explore how this works.

All calculators must retain numbers and operations in memory during calculations. (If this were not true, the calculator would "forget" the 3 when you sought to add 4 to it in the calculation 3 + 4.) To accomplish this RPN calculators have what is called a stack.* (This is sometimes called an automatic memory stack or an operational stack.)

The calculator display is the "bottom" register of the stack. Above it are additional registers. Here is a four register stack:

REGISTER NAME		
T	0	
Z	0	
Y	0	
X	0	DISPLAY

The stack registers are arbitrarily named T, X, Y, Z (the display register), as shown.

As a number is entered in the stack it pushes other numbers up. When an operation is performed the stack (usually) moves down.

EXAMPLE 1. Add 23 and 41

KEYS		STACK	
	T	0	
	Z	0	Step 1. When 23 is keyed, it enters the X-register in the stack.
$\boxed{2}$ $\boxed{3}$	Y	0	
	X	23	DISPLAY

---

*In fact all calculators have similar stacks. On algebraic logic calculators, for example, the $\boxed{(}$ key or even the $\boxed{+}$ key activate a stack. Because the stack plays a greater role in RPN, it is considered here in more detail.

ENT

T	0	
Z	0	
Y	23	
X	23	DISPLAY

Step 2. When ENTER is keyed, the X-register is copied into the Y-register. (Y and Z registers also move up one level.)

4 1

T	0	
Z	0	
Y	23	
X	41	DISPLAY

Step 3. When 41 is keyed, it REPLACES the contents of the X-register.

+

T	0	
Z	0	
Y	0	
X	64	DISPLAY

Step 4. When + is keyed, it adds the X and Y registers. (T and Z registers also move down one level and the T-register becomes 0.)

The power of the ENTER key and the stack will be shown through a second example, a calculation that was a problem for us in AOS-memory and (AOS).

EXAMPLE 2. Calculate $\frac{36.2}{25.8 - 28.3}$

KEYS		STACK	
	T	0	
	Z	0	
3 6 . 2	Y	0	
	X	36.2	DISPLAY

Step 1. Key in 36,2. It appears in the X-register display.

ENT

T	0	
Z	0	
Y	36.2	
X	36.2	DISPLAY

Step 2. ENTER copies X into Y.

$$\frac{36.2}{25.8 - 28.3}$$

2 5 · 8

T	0	
Z	0	
Y	36.2	
X	25.8	DISPLAY

Step 3. The 25.8 replaces 36.2 in the X-register display.

ENT

T	0	
Z	36.2	
Y	25.8	
X	25.8	DISPLAY

Step 4. ENTER copies X into Y and moves Y to Z. This second ENTER key allows us to calculate the denominator separately.

2 8 · 3

T	0	
Z	36.2	
Y	25.8	
X	28.3	DISPLAY

Step 5. The 28.3 replaces 25.8 in the X-register display. Now all numbers are in the stack.

-

T	0	
Z	0	
Y	36.2	
X	-2.5	DISPLAY

Step 6. The contents of the X-register is subtracted from the Y-register. Z moves down to Y. The X-register now displays -2.5 = 25.8 - 28.3, the value of the denominator of the fraction being computed.

$$\frac{36.2}{25.8 - 28.3} = \frac{36.2}{-2.5}$$

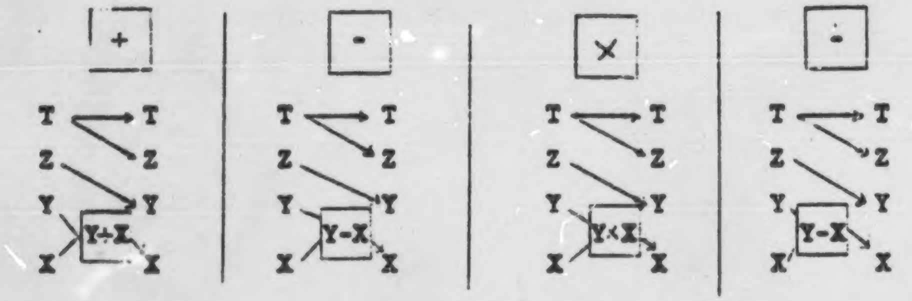
÷

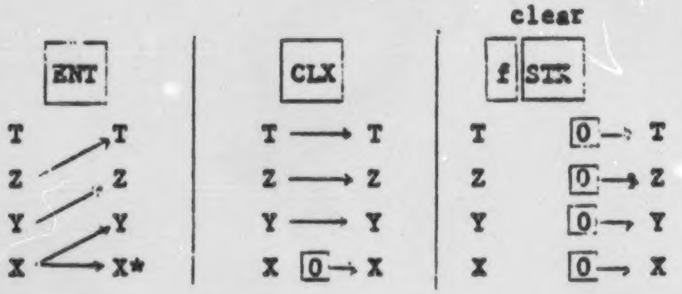
T	0	
Z	0	
Y	0	
X	-14.48	DISPLAY

Step 7. The Y register is divided by the X-register and the answer displayed.

$$\frac{36.2}{25.8 - 28.3} = -14.48$$

The following diagrams will show how the registers change when various keys are depressed.





A final calculator logic which we consider only briefly is called Arithmetic logic. Arithmetic logic is like RPN for addition and subtraction and like algebraic logic for multiplication and division. The easiest way to identify Arithmetic logic calculators is by the combined function keys $\boxed{+/=}$ and $\boxed{-/=}$.

Many business calculators operate with Arithmetic logic. We will not refer to it again in this text.

---

*Recall that if a number is keyed next it will replace this.
In RPN logic, use the ENTER key (for binary operations)
(1) after the first number in a calculation
(2) after the first number in a sub-calculation (the denominator of a fraction on a calculation that would be placed in parentheses.)

Exercise Set 1.4

In each of the following exercises, the stack is shown as it was before the key is depressed. Show what the stack will be after the given key is depressed.

1.		2.		3.		4.	
1		1		8		1	
2	$\boxed{+}$	2	$\boxed{\div}$	0	$\boxed{\times}$	2	$\boxed{\text{ENT}}$
3		3		2		3	
4		4		2		4	

5.		6.		7.		8.	
1		1		1		1	
2	$\boxed{5}$	2	$\boxed{\text{CLX}}$	2	clear	1	$\boxed{\text{CHS}}$
3	BE	3		3	$\boxed{f}\boxed{\text{STK}}$	1	
4	CAREFUL	4		4		1	

9.		10.		11.		12.	
1		1		0		35	
2	$\boxed{-}$	2	$\boxed{\div}$	0	$\boxed{\text{ENT}}$	0	$\boxed{\text{ENT}}$
3		3	WHAT DO YOU	0		0	
4		0	THINK?	35		0	

In the following exercises, show what the stack will be after each key is depressed.

13. $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENT}}$ $\boxed{3}$ $\boxed{+}$

14. $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENT}}$ $\boxed{\times}$

15. $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\times}$

16. $\boxed{\text{CLR}}$ $\boxed{3}$ $\boxed{\text{ENT}}$ $\boxed{\text{ENT}}$ $\boxed{+}$ $\boxed{\times}$

17. $\boxed{\text{CLR}}$ $\boxed{2}$ $\boxed{3}$ $\boxed{\text{ENT}}$ $\boxed{5}$ $\boxed{\div}$

18. $\boxed{\text{CLR}}$ $\boxed{5}$ $\boxed{\text{ENT}}$ $\boxed{4}$ $\boxed{\text{ENT}}$ $\boxed{+}$ $\boxed{\div}$

19. Express in algebraic form the calculation carried out in exercises 14 - 18. (For example, exercise 13 is 5 + 3.)

Give the RPN keystrokes for the following computations. Then calculate.

20. $(2 + 3)4$

21. $\frac{2}{3} + 4$

22. $4 + \frac{2}{3}$ (Hint: $\boxed{4}$ $\boxed{\text{ENT}}$ $\boxed{2}$ $\boxed{\text{ENT}}$ $\boxed{3}$ $\boxed{}$ $\boxed{}$)

23. $(2 + 3)\ (4 + 5)$

24. $\frac{2}{3} + \frac{4}{5}$

25. $\frac{2 + 3}{4 + 5}$

26. $(2 + 3)\ (4 + 5)(6 + 7)$

27. Show a second way to calculate exercise 26.

28. Give a keystroke sequence that will fill the stack in the following way

T	6	
Z	7	
Y	8	
X	8	DISPLAY

29. Recalculate exercises 16 and 18 with the stack at the beginning of the calculation in the form of exercise 28 and omitting the $\boxed{\text{CLR}}$ key. This exercise should show you that IT IS NOT NECESSARY TO CLEAR THE STACK IN ORDER TO CARRY OUT MOST CALCULATIONS.

30. Use an RPN calculator to compute the answer to exercise 17 of Section 1.3 (on page 1.3 - 4).

1.5 Other Calculator Keys

We have studied differences between AOS-memory, (AOS) and RPN logics. Most of these differences apply to binary operations, that is operations that combine two elements into one. Addition, subtraction, multiplication and division are the commonest binary operations of arithmetic.

The following operations are unary operations, that is operations that need only one element to process.

$\sqrt{x}$

x^2

$1/x$

10^x

sine

cosine

tangent

CHS

INT

FRACT

ABS

(We will introduce other unary operations such as $\log x$, $\ln x$, and e^x, later.)

All calculators process unary operations by RPN! The x-value is keyed into the calculator and the function key is pressed.

EXAMPLE 1. Calculate $\sqrt{51}$

Keystroke sequence $\boxed{5}$ $\boxed{1}$ $\boxed{\sqrt{x}}$

Answer: 7.1414

EXAMPLE 2. Calculate $\sin 30^\circ$

Keystroke sequence* $\boxed{3}$ $\boxed{0}$ $\boxed{\sin}$

Answer: .5

EXAMPLE 3. Find the reciprocal of 10^2

Keyboard sequence 1: $\boxed{1}$ $\boxed{0}$ $\boxed{x^2}$ $\boxed{1/x}$

Keyboard sequence 2: $\boxed{2}$ $\boxed{10^x}$ $\boxed{1/x}$

Answer: .01

---

*Most calculators assume input to trigonometric functions to be in degrees. We will also unless otherwise mentioned.

In all cases these function keys operate on the number in the display register. Note that it is not necessary to depress the ENTER key on an RPN-calculator before using these functions. For any unary function the stack diagram is:

$$\begin{tikzcd} & \boxed{f} & \\ T \arrow[rr] & & T \\ Z \arrow[rr] & & Z \\ Y \arrow[rr] & & Y \\ X & f(x) \arrow[r] & X \end{tikzcd}$$

The unary functions INT, FRACT and ABS will be considered in the exercises.

One important function that does differ between AOS and RPN calculators is exponentiation (raising to a power). This is a binary operation because

$$p^q$$

requires the two input elements p and q.

EXAMPLE 4. Compute 7^4

AOS Keystroke sequence: $\boxed{7}\ \boxed{y^x}\ \boxed{4}\ \boxed{=}$

RPN Keystroke sequence: $\boxed{7}\ \boxed{\text{ENT}}\ \boxed{4}\ \boxed{y^x}$

Answer: 2401

When using this $\boxed{y^x}$ key, you will meet for the first time the fact that the calculator sometimes produces only approximate answers. In the calculation of 7^4, for example, a calculator may display the answer 2400.9993. Now we know that 7^4 is an integer and we can find it exactly my multiplying 7 × 7 × 7 × 7 to get 2401. The error (of .0007 in this case) is introduced by the logarithmic processing used by the calculator

$\boxed{y^x}$ key. We will study this later. For now it is usually enough to round off such answers to the nearest integer.

Exercise Set 1.5

Without using a calculator give the display produced by the following keystroke sequences. Check your results by calculator. Start each check by clearing the display.

1. $\boxed{5}\ \boxed{x^2}$
2. $\boxed{9}\ \boxed{\sqrt{x}}$
3. $\boxed{4}\ \boxed{1/x}$
4. $\boxed{3}\ \boxed{10^x}$
5. $\boxed{8}\ \boxed{\text{CHS}}$
6. $\boxed{0}\ \boxed{1/x}$
7.* $\boxed{5}\ \boxed{\text{ENT}}\ \boxed{x^2}$ What does the stack look like after this sequence?
8. $\boxed{5}\ \boxed{\text{ENT}}\ \boxed{x^2}\ \boxed{+}$
9. $\boxed{4}\ \boxed{10^x}\ \boxed{\sqrt{x}}$
10. $\boxed{7}\ \boxed{\text{CHS}}\ \boxed{x^2}$
11. $\boxed{2}\ \boxed{x^2}\ \boxed{1/x}$
12. $\boxed{2}\ \boxed{1/x}\ \boxed{x^2}$
13. $\boxed{5}\ \boxed{1/x}\ \boxed{\times}$
14. $\boxed{5}\ \boxed{\text{ENT}}\ \boxed{1/x}\ \boxed{\times}$
15. $\boxed{1}\ \boxed{10^x}$
16. $\boxed{(}\ \boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{)}\ \boxed{10^n}$
17. $\boxed{9}\ \boxed{+}\ \boxed{7}\ \boxed{=}\ \boxed{\sqrt{x}}$

By applying the functions to various values, determine what the following keys do. Be sure to include values like 7.65, -3, -3.72.

18. INT
19. FRACT
20. ABS

---

*Note that the keys will tell which logic is used.

Calculate each of the following. Check your result against the answer given.

Example: $5 + \sqrt{7}$

AOS keystroke*: $\boxed{5}$ $\boxed{+}$ $\boxed{7}$ $\boxed{\sqrt{x}}$ $\boxed{=}$

RPN keystroke: $\boxed{5}$ $\boxed{\text{ENT}}$ $\boxed{7}$ $\boxed{\sqrt{x}}$ $\boxed{+}$

Answer: 7.6457

21. 8^5 Ans. 32768

22. 1.23^3 Ans. 1.8609

23. $\frac{1}{16} + \frac{1}{7}$ Ans. 0.2054

24. $\frac{1}{16 + 7}$ Ans. 0.0435

25. $10^5 - 5^7$ Ans. 21875 (On some algebraic calculators you may find it necessary to use parentheses around 5^7.)

Now try your hand at the following monsters:

26. $\frac{\sqrt{35} \sin 45^\circ}{34^3}$ Note: the numerator is a product. Ans. 0.0001

27. $\text{INT}(10^{\tan 60^\circ})$ Ans. 53

28. $\frac{\sqrt{3.7} + \cos 10^\circ}{.13^3 - \frac{1}{27}}$ Ans. -83.477125

The following two exercises provide useful short-cuts for computation:

29. Sometimes the wrong number appears in the display. For example, when you wish to calculate $a - b$, b may already be displayed. How could you complete the calculation without starting all over?

30. How can you calculate $\frac{a}{b}$ starting with b in the display?

---

*Some AOS calculators will not accept this calculation. On them it must be rendered to $\sqrt{7} + 5$, or it must be calculated as $5 + (\sqrt{7})$.

1.6 Problem Solving with a Calculator

With the power your calculator gives you, you may now attack with confidence and solve some complicated problems. You will need paper and pencil only to record notes and answers. As you will see, however, the calculator does not substitute for thinking. You are still in charge. You will need to

- organize calculations so that you can carry them out on your calculator

and if your problem is one related to measurement

- determine units for the answer and
- determine accuracy

In this section we will not deal with the latter two important questions. We will continue to report answers to four digit (rounded), accuracy.*

EXAMPLE 1. A simplified formula for artillery range is

$$R = \frac{Vo^2 \sin A \cos A}{9.8}$$

Find the numerical value (without units) of R when

$$Vo = 31 \text{ and } A = 30^\circ.$$

SOLUTION. Substituting

$$R = \frac{31^2 \sin 30^\circ \cos 30^\circ}{9.8}$$

Calculation yields 42.4618

---

* Some calculators truncate answers rather than round answers. Truncated means that the rest of the answer is cut off. Thus 683.29587 truncated to six digits is 683.295, the .00087 merely dropped. This is often called "rounding down". You should test your calculator to see how it rounds. Use quotients like 2/3, 5/33, and 50/33.

Such a calculation is important but straightforward. Others require an experimental approach.

EXAMPLE 2. In EXAMPLE 1 we might wonder what angle A makes R largest. (What angle of elevation yields longest range?)

SOLUTION. We need only consider the product

sin A cos A

Trying values yields

A	sin A cos A
30°	0.4330
40°	0.4924
50°	0.4924

This suggests trying A = 45° (why?)

45°	.5

Trying other values suggests that this is the best we can get in the range 0° to 90°.

Often it simplifies computation to use storage capacity of your calculator to evaluate expressions in which letters appear more than once. In the following example, we assume a calculator that has at least two storage registers R1 and R2. To store 5 in R1 and 6.3 in R2 the following keystroke sequence could be used:

$$\boxed{5}\quad\boxed{\text{STO}}\quad\boxed{1}\quad\boxed{6}\quad\boxed{\bullet}\quad\boxed{3}\quad\boxed{\text{STO}}\quad\boxed{2}$$

This sequence of keys is appropriate for most algebraic or RPN calculators.*

---

*On some calculators each register may have a two-digit designation. In that case to store 5 in R01 would be keyed

$$\boxed{5}\quad\boxed{\text{STO}}\quad\boxed{0}\quad\boxed{1}\ .$$

To recall the number in R1, you need only press

$\boxed{\text{RCL}}$ $\boxed{1}$

and in this case the 5 will reappear in the display.

EXAMPLE 3. Evaluate $x^3 + 3x^2y + 3xy^2 + y^3$ for $x = 3.7$ and $y = 8.6$.

SOLUTION: If you attack this problem directly you will be keying 3.7 and 8.6 each several times. You can save some of these keystrokes by first storing x and y. Follow the program for the kind of calculator you use.

$\boxed{3}$ $\boxed{\cdot}$ $\boxed{7}$ $\boxed{\text{STO}}$ $\boxed{1}$ $\boxed{8}$ $\boxed{\cdot}$ $\boxed{6}$ $\boxed{\text{STO}}$ $\boxed{2}$

or

Algebraic	representing	RPN
$\boxed{\text{RCL}}$ $\boxed{1}$ $\boxed{y^x}$* $\boxed{3}$ $\boxed{+}$	x^3	$\boxed{\text{RCL}}$ $\boxed{1}$ $\boxed{3}$ $\boxed{y^x}$*
$\boxed{(}$ $\boxed{\text{RCL}}$ $\boxed{1}$ $\boxed{y^x}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{3}$ $\boxed{)}$ $\boxed{+}$	$3x^2y$	$\boxed{\text{RCL}}$ $\boxed{1}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{3}$ $\boxed{\times}$ $\boxed{+}$

---

*Beware! Do not confuse the X and Y registers with the x and y in the polynomial. The $\boxed{y^x}$ key operates on numbers in the appropriate calculator registers.

Algebraic	representing	RPN
$\boxed{(}$ $\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{2}$	$3xy^2$	$\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{\text{RCL}}$
$\boxed{\times}$ $\boxed{\text{RCL}}$ $\boxed{1}$ $\boxed{\times}$ $\boxed{3}$		$\boxed{1}$ $\boxed{\times}$ $\boxed{3}$ $\boxed{\times}$
$\boxed{)}$		$\boxed{+}$
$\boxed{+}$		
$\boxed{(}$ $\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{y^x}$ $\boxed{3}$	y^3	$\boxed{\text{RCL}}$ $\boxed{2}$ $\boxed{3}$ $\boxed{y^x}$
$\boxed{)}$		$\boxed{+}$
$\boxed{-}$		

You should reach the value 1860.867.

In solving complex problems like these you will need to be very careful. Here are some suggestions which may help:

(1) Think through your computation before you start to key numbers into the calculator.

(2) Try to organize your computation in parts such as terms of a polynomial or the numerator and denominator of a fraction.

(3) If you feel you will be lost computing the answer to a complex problem in one series of keystrokes, take it part by part, recording partial answers. You may then combine these into a final solution.

(4) Sometimes (as you will see in the exercises) algebraic simplification of an expression to be evaluated will also simplify computation.

Exercise Set 1.6

Evaluate the formula $R = \frac{Vo^2 \sin A \cos A}{9.8}$ for R using the given values of Vo and A.

1. $Vo = 200, A = 40^\circ$
2. $Vo = 100, A = 40^\circ$
3. $Vo = 100, A = 50^\circ$
4. $Vo = 375, A = 90^\circ$

(You may wish to think about the results of exercises 1-4 as they relate to the physics of projectile range.)

Using one of the two conversion formulas for Celsius and Fahrenheit temperatures,

$$C = \frac{5}{9}(F - 32) \qquad F = \frac{9}{5}C + 32$$

answer the following:

5. $C = 100^\circ$, find F
6. Convert $32^\circ F$ to C.
7. Change $68^\circ F$ to C
8. Change $98.6^\circ F$* to C.
9. Find by experimenting when F and C are the same.
10. Now check your answer to exercise 9 by algebra. (Set F and C each equal to x in one of the two formulas and solve for x.)

A formula for triangle area that you will be able to derive later is

$$A = (s_1 \cdot s_3 \cdot \sin A_2) \div 2$$

---

*Average human body temperature

Find A, given the following values:

11. $s_1 = 2$, $s_3 = 5$, $A_2 = 50^\circ$
12. $s_1 = 10$, $s_3 = 8$, $A_2 = 60^\circ$
13. $s_1 = 3.72$, $s_3 = 5.8$, $A_2 = 38^\circ$
14. $s_1 = 147.3$, $s_3 = 62.1$, $A_2 = 72^\circ$

If an object is h meters above the ground, the time, t, in seconds, that it takes to fall to the ground is given by the formula

$$t = \sqrt{\frac{2h}{9.8}}$$

Find t when:

15. $h = 147.2$
16. $h = 3472.13$
17. $h = 1.23$
18. $h = 43.278$
19. Solve the formula for h and use your new formula to find h when $t = 10$.

In a right triangle whose legs are a and b and whose hypotenuse is c, you know that $c = \sqrt{a^2 + b^2}$.

Find c when:

20. $a = 5$ and $b = 4$
21. $a = 10.35$ and $b = 15.72$
22. $a = 10.3$ and $b = 11.7$
23. $a = 2.3$ and $b = 18.9$

Evaluate when $x = 3.7$ and $y = 8.6$. Store these values for x and y.

24. $x^2 + 2xy + y^2$
25. $(x + y)^2$
26. $x^2 + y^2$
27. $(x + y)^3$

28. What identity to your answers in exercises 24 and 25 support?

29. What do your answers in 24, 25 and 26 suggest?

30. What identity do your answers to exercise 27 and the answer to example 3 on pages 1.6 - 3 and 4 suggest?

1.7 Programming Functions: 1

In working the exercises of Section 1.6 you should have found the calculations repetitious. You were following similar routines over and over, with only the numbers different. In this section we will develop a short-cut to reduce such work.

EXAMPLE 1. Give a keystroke routine that will start with a given value of h and calculate t by the formula

$$t = \sqrt{\frac{2h}{9.8}}$$

SOLUTION:

Algebraic	RPN
key in h, then	key in h, then
$\boxed{\times}$ $\boxed{2}$ $\boxed{\div}$ $\boxed{9}$ $\boxed{.}$ $\boxed{8}$ $\boxed{=}$ $\boxed{\sqrt{x}}$	$\boxed{\text{ENT}}$ $\boxed{2}$ $\boxed{\times}$ $\boxed{9}$ $\boxed{.}$ $\boxed{8}$ $\boxed{\div}$ $\boxed{\sqrt{x}}$

Notice that once the keystrokes have been worked out it requires no knowledge of the function to follow them. With these instructions you could give your homework exercises to an elementary school aged sister or brother to calculate for you. For example, given the h value 10, they would key $\boxed{1}$ $\boxed{0}$ and then the keystrokes for your calculator, giving the resulting t value, 1.4286.

Still better, you can assign this routine to your programmable calculator. Here are the general steps you can use to accomplish this:

LRN - Learn
R/S - Run-Stop
RST - Reset

1. Set your calculator to record a program.
2. Key into your calculator the calculation steps (along with any instruction steps necessary to your particular calculator.)
3. Set your calculator back to calculating mode.
4. Enter your given data.
5. Run the program.

For additional exercises of the same type, you then merely repeat steps 4 and 5.

Each of the many calculator models operates differently so it is not possible to list all the special instructions required to carry out the five step routine we have just given. Because they suggest the kinds of special differences you will meet on calculators, we offer two examples here. You should study them to see their form, but you should concentrate on the specific routines for the calculator you will be using. Recall that we are programming the calculation

$$\sqrt{\frac{2h}{9.8}}$$

TI - 57

1. OFF - ON — This clears the calculator of previous programs.

LRN — LRN sets this calculator to record the program. The display is 000 00

2. × 2 ÷ 9 . 8 = √x — These are the calculation steps. (See the SOLUTION to EXAMPLE 1 on page 1.7 - 1.) As you depress each key the calculator display will record the step number. 001 00 up to 008 00.

PRGM - Program

	R/S	This key is required to stop the program and display the result of the calculation.
	RST	RST instructs the calculator to go to step 00 in the program
3.	LRN	This key now returns the calculator to normal operation. The display is 0.
4.	RST	RST sets the calculator to begin the program.
5.	Key in h.	
6.	R/S	R/S then activates the program. When the calculator stops the display will give the t value.

To find additional pairs (h,t), repeat steps 5 and 6.

HP - 33

1.	OFF - ON	This clears the calculator of previous programs.
	PRGM	PRGM sets the calculator to record your program steps. The display is 00 .
2.	ENTER↑ 2 × 9 · 8 ÷ f* √X	These are the calculation steps. As you depress each key the display records the step number and the location (row - column) of the key(s) depressed. For example, after ENTER is pressed 01 31 is displayed: 01 is the step number, 31 the location (row 3, key 1) of ENTER on the keyboard. Note how the last two keys are merged into one step 08 14 02. This saves program steps.

---

*On this calculator, as on many others, many keys have two or even three roles. Here the yellow f key assigns the second role √x to the 2 key. The blue g key would have assigned x^2 to the same key.

SST - single step

3. RUN*
$\boxed{\text{SST}}$ $\boxed{\text{R/S}}$
The calculator is now returned to normal operation, the display is 0.00
The R/S key sets the program back to step 00.

4. Key in h

5. $\boxed{\text{R/S}}$
This activates the program. On this calculator we did not have to key another $\boxed{\text{R/S}}$ into the program because all unused program steps are pre-loaded with steps that return the program to step 00 and stop it there.

To find additional pairs (h,t) repeat steps 4 and 5.

You should familiarize yourself with the procedures for entering and running programs, but the more important task is developing programs. Here are some suggestions about how to do this:

1. Remember that the program merely records what you would have done in a calculation that is not programmed.
2. Think of your calculation as always starting from the value(s) that change in the computation. (In the example this was h.)
3. Key into the program the steps following (and not including) the step that keys your starting value (see suggestion 2) into the display. (On an RPN calculator don't forget ENTER when it is necessary.)
4. Be sure, if your calculator requires it, to complete your program with $\boxed{\text{R/S}}$ so your calculator will stop to display the results.

Exercise Set 1.7

Key into your calculator a program to find t, given h, by the formula

$$t = \sqrt{\frac{2h}{9.8}}$$

---

*The $\boxed{\text{SST}}$ key steps out of the program so that $\boxed{\text{R/S}}$ does not calculate the last step of the program. This prevents occasional Error messages as when the last key is $\boxed{\div}$.

Then calculate t for the following h values

1. 15
2. 100
3. 1000
4. 10,000
5. 8840 (m in ht. of Mt. Everest)
6. 1609 (m in a mile)
7. Develop a keystroke sequence to change any Fahrenheit temperature into Celsius by the formula

$$C = \frac{5}{9}(F-32)$$

(Don't forget to start your calculation from F.)

Program the calculation of exercise 7 and use it to convert the following temperatures to Celsius:

8. 0° F
9. 90° F
10. 50° F
11. -40° F
12. By experimenting, find when $F = 2C$, that is when Fahrenheit temperature is twice Celsius temperature.
13. The sales tax in Erie County, New York is 7%. Develop a keystroke sequence that will calculate the amount of this sales tax. (Do not bother with rounding your answer.)

Program the calculation of exercise 13 and use it to determine sales tax on the following purchases:

14. $500
15. $45.32
16. $299.95
17. $2.79
18. By experimenting, find a purchase price that will give a sales tax of $1.00.

1.8 Programming Functions: 2

In Section 1.7 you learned to program your calculator so that it would carry out computation routines by a single keystroke. In that section you were restricted to single input-single output routines. Now in this section you will learn how to handle more than one input or output.

The key to this problem and the key to press is:

R/S

This powerful key plays the following important roles:

1. When the calculator is in operating mode, it either starts a program if the calculator is idle or stops a running program.
2. When it is keyed into a program it stops the program either to receive information or to give information.

We will consider how this works by means of examples.

EXAMPLE 1 Develop and run a program to evaluate c for entered values of a and b, in the formula

$$c = \sqrt{a^2 + b^2}$$

SOLUTION 1.

by TI -57		by HP - 33
OFF - ON		OFF - ON
LRN	a would be keyed before the program started.	PRGM
x^2 +		g x^2
* R/S	Here the calculator is stopped to receive b.	* R/S

by TI -57	by HP - 33
x^2	g x^2
=	+
$\sqrt{x}$	f $\sqrt{x}$
R/S	
RST	
LRN	RUN

Here is how these programs would be run for a = 5, b = 12.

RST	Resetting the proram to 0	SST R/S
5 R/S	Enter 5 and start the program	5 R/S

The first part of the program runs until it reaches

R/S at the step marked * in the program. It stops with the display reading 25.

1 2 R/S	Enter 12 and restart.	1 2 R/S

The calculators will now complete their programs and display the c value 13.

You will develop other ways of carrying out this kind of multiple input program in the exercises. We now consider a problem involving multiple output.

EXAMPLE 2. Develop a program that will calculate and display sales tax (at 7%) and then total cost for given purchase prices.

TI - 57

OFF - ON* LRN

STO 1	Stores purchase price
× . 0 7 =	Calculates sales tax
R/S	Stops to display tax
+ RCL 1 =	Adds on purchase price to give total cost.
R/S	
RST	
LRN	
RST	

Running the program for a $92 purchase

9 2 R/S

The calculator runs to the first R/S , and stops there to display the sales tax $6.44

R/S The calculator completes the program and displays the total cost $98.44

---

*On this calculator there are other ways to clear programs and reset the program to 0, but we adopt this simple procedure. In fact, new programs may be keyed right "over" old ones for the new steps replace the old.

HP - 33

Storage could be used as in the TI - 57 solution but instead we utilize the operating stack to solve this problem.

OFF - ON*, PRGM

ENTER ENTER	Now the purchase price is in Y, and Z registers.
. 0 7 ×	Now sales tax is in X, purchase price in Y.
R/S	Stop to display sales tax
+	Adds sales tax and purchase price.

RUN

Running the program for a $92 purchase

SST R/S	Resets to 00
9 2 R/S	

Now the calculator displays the sales tax $6.44

R/S

The calculator completes the program and displays the total cost $98.44.

---

*On this calculator also there are again other ways to clear programs and reset the program to 0, but we adopt this simple procedure. In fact, new programs may be keyed right "over" old ones for the new steps replace the old.

Exercise Set 1.8

1 - 4 Program EXAMPLE 1 of page 1.8 - 1 into your calculator and use this program to find $\underline{c}$ for the following:

1. $\underline{a} = 23$, $\underline{b} = 264$
2. $\underline{a} = 45$, $\underline{b} = 24$
3. $\underline{a} = 45$, $\underline{b} = 336$
4. $\underline{a} = 7$, $\underline{b} = 24$

5 - 9 For $\underline{a} = 45$, there are five other values of b that result in Pythagorean triples, that is results for $\underline{a}$, $\underline{b}$, and $\underline{c}$ all in integers.

Find the $\underline{b}$ and $\underline{c}$ that completes the ($\underline{a} = 45$) triple for $\underline{b}$ in each of the following ranges:

5. $25 \leq b \leq 30$
6. $60 \leq b \leq 65$
7. $105 \leq b \leq 110$
8. $195 \leq b \leq 200$
9. $1010 \leq b \leq 1015$

10 - 15 Program EXAMPLE 2 of page 1.8 - 3 into your calculator and use this program to find sales tax and total cost for the following purchase prices. *

10. \$34.95
11. \$1.67
12. \$2995
13. \$632.50
14. \$99.95
15. \$100

16. How could you modify the program of EXAMPLE 2 if sales tax went up to 8%? Clearly you can change and reenter the entire program, but you may wish to experiment with calculator keys in $\boxed{\text{LRN}}$ or PRGM mode to make the necessary key change. You will need to determine how the following keys work on your calculator:

---

*On calculators that display four decimal digits (like the HP- 33) you need to exercise care here. Such calculators probably do not round up but either round down (truncate) or round to the nearest value. Your best procedure is to reset such calculators to display more decimal digits.

On the HP - 33, for example, to set three decimal places in the display press $\boxed{\text{f}}$ $\boxed{\text{FIX}}$ $\boxed{3}$.

SST	single step
BST	back step

and on the TI - 57

2^{nd}	INS	insert
2^{nd}	DEL	delete

17. (for algebraic calculators only) In EXAMPLE 2 you had to store the purchase price because it is lost when you calculate sales tax. Show how you can avoid storage by calculating total cost from the sales tax. (HINT: If purchase price is p, sales tax is .07p and total cost is 1.07p. Determine the number you must multiply .07p by to get 1.07p.)

18. Suppose you were a householder in an area where different communities in which you shopped charged different sales taxes. (This is fairly common near state or even county boundaries.) Develop a sales tax - total cost program so that you can enter list price and then sales tax rate to produce sales tax and total cost. (Hint: an easy way to do this is to use program storage.) Use your program to complete the following table:

	list price	tax rate	tax	cost
suit	$137.95	6%		
overcoat	84.50	8%		
shoes	31.45	7%		
hat	18.50	6%		
TOTALS	-----	------		

1.9 Chapter 1 Test

(1 - 4) Evaluate each of the following keystroke sequences. Do not use your calculator.

1) $\boxed{2}\ \boxed{x^2}\ \boxed{\frac{1}{x}}$

2) $\boxed{4}\ \boxed{+/-}\ \boxed{\sqrt{x}}$

3) $\boxed{2}\ \boxed{\text{STO}}\ \boxed{1}\ \boxed{5}\ \boxed{\text{RCL}}\ \boxed{1}$

4) $\boxed{2}\ \boxed{10^x}$

(5 - 8) Give a keystroke sequence that you could use on your calculator to evaluate each of the following expressions.

5) $5 + \frac{6}{7}$

6) $(2 + 3)(4 + 7)$

7) $\sqrt{6 \sin 37^\circ}$

8) $\frac{2 + 3}{4(7)}$

(9 - 10) Evaluate each of the following expressions

9) $2 \div 10 \times 5$

10) $3 + 2 \times 5$

(11 - 12) Evaluate each of the following expressions. Express your answer correct to four decimal places.

11) $(3.5 + 8.32)^2\ (4.3 - 5.31)^2$

12) $\frac{5.7\,(6.5 - 4.8)}{2.3 - 5.8}$

(13 - 14) Evaluate each of the following expressions. Round your answer to the nearest hundredth.

13) $\frac{\sqrt{8.7} + \sin 37.5^\circ}{1.5^2 - \frac{1}{3.2}}$

(14) The area of a triangle with sides a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a+b+c)$$

Find the area of a triangle whose sides are 35.41 cm., 28.23 cm. and 24.68 cm.

15) The time needed to complete one period of a pendulum is given by the formula

$$t = \pi\sqrt{\frac{\ell}{980}}$$

where ℓ represents the length of the pendulum

Find the value for ℓ, correct to the nearest tenth, that makes t closest to 4 seconds.

(16 - 18) Choose two of the following three problems.

16) A manufacturing plant has a capacity of 25 articles per week. Experience has shown that depending on the number of articles (n) manufactured per week, the price of an article (p) will be sold at a variable price where $p = 110 - 2n$. The cost of manufacturing n articles is $600 + 10n + n^2$

a) Express in terms of n and p, the total amount of money received (gross income) in one week for selling n articles.

b) Express this weekly gross income in terms of n only.

c) Express the total weekly profit or loss (net income) for the company in terms of n only.

d) Find the total profit if only 8 items are produced per week.

e) Write a program that can be used to determine the weekly profit for any value of n.

f) How many articles should be made each week to give the largest profit?

17) a) Write a program for finding the length of a leg (x) of a right triangle given the lengths of the hypotenuse (z) and the other leg (y).

b) List the keystrokes necessary in the run mode to find the leg (x).

c) Load your program into your calculator and find x, to the nearest hundredth when

(i) $z = 70.00$ and $y = 66.18$

(ii) $z = 24.25$ and $y = 18.75$

18) As already mentioned in exercise (14) the area of a triangle with sides a, b, and c is given by the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{1}{2}(a+b+c)$$

a) Write a program that will find the area of a triangle given the lengths of the three sides. Store a in register 1, b in register 2, c in register 3 and s in register 4.

b) Load your program into your calculator and find the area of each of the following triangles to the nearest integer

(i) $a = 88$, $b = 72$, $c = 108$

(ii) $a = 2$, $b = 3$, $c = 5$

(iii) Explain your answer to (ii).

CHAPTER 2. EXPONENTS AND LOGARITHMS

In this section the theory of exponents is extended and a different kind of exponent, the logarithm, is introduced.

2.1 Positive Integer Exponents

An important role of exponents is abbreviation, shortening expressions. Just as we use multiplication to abbreviate addition when we replace a + a + a + a + a by 5a, so we abbreviate multiplication when we replace $a \cdot a \cdot a \cdot a \cdot a$ by a^5. In general:

DEFINITION 2.1.1 For real x and n a positive integer

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

In particular, $x^1 = x$.

When we write p^q, p is called the base and q the exponent or power. The word power is also used for the whole expression p^q. Thus we speak of 4, 8, 16, ... as the powers of 2 just as we would 4, 6, 8, 10,... as the multiples of 2.

Notation:

$$\underset{\text{base}}{p}^{\overset{\text{power or exponent}}{q}} = \underset{\text{power}}{r}$$

Exponential statements are read:

3^2 - three squared

2^3 - two cubed

5^4 - five to the fourth power

6^n - six to the n'th power or six to the n

a^b - a to the b or a to the b power

The following six theorems are often called the Laws of Exponents. We will see later that they apply in much broader circumstances: here they apply only to positive integral exponents.

The Laws of Exponents

For a and b positive integers, x and y real,

(1) $x^a \cdot x^b = x^{a+b}$

(2) $x^a \div x^b = x^{a-b}$, $a > b$

(3) $(x^a)^b = x^{ab}$

(4) $\sqrt[b]{x^a} = x^{\frac{a}{b}}$, a a multiple of b, $x > 0$

(5) $(xy)^a = x^a y^a$

(6) $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ *

(7) $\sqrt[a]{xy} = \sqrt[a]{x}\,\sqrt[a]{y}$, $x > 0$, $y > 0$

Proofs of these theorems are so straight forward that many students are confused by them. They merely involve applying definition 2.1.1 and counting the number of factors. We will display two proofs and leave the others for the exercises:

EXAMPLE 1. Prove: For a and b positive integers, x real,

$$(x^a)^b = x^{ab}$$

$$(x^a)^b = \underbrace{x^a x^a x^a \ldots x^a}_{b \text{ factors}} \qquad \text{Def. 2.1.1}$$

$$= \underbrace{\underbrace{x \cdot x \cdot \ldots x}_{a \text{ factor}} \cdot \underbrace{x \cdot x \ldots x}_{a \text{ factor}} \cdot \underbrace{x \cdot x \ldots x}_{a \text{ factor}} \ldots \underbrace{x \cdot x \ldots x}_{a \text{ factor}}}_{b \text{ (groups of) factors}}$$

---

*Some authors apply the restriction $y \neq 0$ here. We choose instead to embed this restriction in the = symbol. Thus "=" carries with it the phrase "for which both members are defined". This is not uncommon usage.

$$= \underbrace{x \cdot x \cdot \ldots \cdot x}_{ab \text{ factors}} \qquad \text{counting}$$

$$= x^{ab} \qquad \text{Def. 2.1.1}$$

Notice that Definition 2.1.1 is used left to right in the first and second step of the proof, right to left in the last, that is as

$$\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}} = x^n$$

Students often forget that definitions (and equations) are symmetric.

EXAMPLE 2. Prove: For a , a positive integer, x and y real,

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

$$\left(\frac{x}{y}\right)^a = \underbrace{\frac{x}{y} \cdot \frac{x}{y} \cdot \ldots \frac{x}{y}}_{a \text{ factors}} \qquad \text{Def. 2.1.1}$$

$$= \frac{\overbrace{x \cdot x \ldots x}^{a \text{ factors}}}{\underbrace{y \cdot y \ldots y}_{a \text{ factors}}} \qquad \text{Using } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

$$= \frac{x^a}{y^a} \qquad \text{Def. 2.1.1 (applied separately in numerator and denominator)}$$

EXAMPLE 3. Simplify $x^3 \cdot x^2$, $2^x \cdot 2^y$, $\frac{3^7}{3^5}$, $(\pi^2)^5$, $\sqrt{x^{10}}$, $\sqrt[a]{6^{3a}}$, $(-2x)^5$, $\left(\frac{2}{x}\right)^3$

$x^3 \cdot x^2 = x^5$ Law (1)

$2^x \cdot 2^y = 2^{x+y}$ Law (1)

$\frac{3^7}{3^5} = 3^2 = 9$ first step by Law (2)

$(\pi^2)^5 = \pi^{10}$ (Law (3)

$\sqrt{x^{10}} = x^{10/2} = x^5$ first step by Law (4)

$\sqrt[a]{6^{3a}} = 6^{\frac{3a}{a}} = 6^3 = 216$ first step by Law (4)

$(-2x)^5 = (-2)^5x^5 = -32x^5$ first step by Law (5)

$(\frac{2}{x})^3 = \frac{2^3}{x^3} = \frac{8}{x^3}$ first step by Law (6)

Exercise Set 2.1

1 - 40 Simplify completely:

(1) $x^6 \cdot x^6$ (2) $2a^2 \cdot 3b^3$

(3) $10w^6 \div 5w^2$ (4) $\frac{60a^{12}}{10a^6}$

(5) $(c^5)^3$ (6) $(2w^3)^3$

(7) $\sqrt[4]{81a^{12}}$ (8) $\sqrt[c]{5^{2c}}$

(9) $(x^2y^3)^4$ (10) $(-2c^3d)^6$

(11) $(\frac{2}{3})^3$ (12) $(\frac{x^2y^5}{x^3})^4$

(13) $(\frac{a^7b}{a^2})^4$ (14) $3x^2 + 4x^2$

(15) $(-2x)^5 \div (2x)^2$ (16) $\sin^3x \cdot \sin^2x$ *

---

*$\sin^n x$ means $\underbrace{\sin x \cdot \sin x \cdot \ldots \cdot \sin x}_{n \text{ factors}}$, thus $\sin^n x = (\sin x)^n$. We will see later that $n \neq -1$.

(17) $2\sin^2 x \cos^2 x + (3\sin x \cos x)^2$

(18) $e^x \cdot e^{x+1}$

(19) $\frac{\pi r^2}{r}$

(20) $2^y \cdot 2^{2y}$

(21) $\frac{25^{17}}{25^{15}}$

(22) $\frac{8^3}{2^3}$

(23) $\sqrt{a^6 b^4}$ *

(24) $(2x + y)^4$

(25) $(x^2 y)^3 \div y^2$

(26) $(\frac{1}{2} t^7)^3$

(27) $(3\frac{1}{4})^3$

(28) $(5^{x+1})^3$

(29) $\frac{(x + 7)^5}{(x + 7)^2}$

(30) $(a + 3)^2 (a + 3)^4$

(31) $\cos^5 x \div \cos^2 x$

(32) $-r^3 \div r$

(33) $(-2y)^4$

(34) $(x^a)^5$

(35) $\sqrt[5]{32b^{20}}$

(36) $\sqrt[n]{x^{2n}}$

(37) $(3 \tan^3 x)^2$

(38) $3^{xa} \div 3^{2a}$

(39) $(x^{2a+1})(x^{3-2a})$

(40) $(2x^3)^{a+2}$

41 - 50 Evaluate:

(41) $(x^2 y^3)^2$ if $x = 3$ and $y = -1$

(42) $5x^3 + 4y^2$ if $x = -1$ and $y = 3$

(43) $(-2ab^2)^3$ if $a = 5$ and $b = 2$

(44) $3a^2 - (5b)^3$ if $a = 2$ and $b = -2$

(45) $\sqrt[3]{8a^{12} b^9}$ if $a = -2$ and $b = -1$

(46) $2^c \cdot 2^{2c}$ if $c = 2$

---

*Recall that a radical with no designated index means the index is 2.

(47) $(x^{4a})(x^{2a+3})^2$ if $x = -1$ and $a = 2$

(48) $5^{2x-1} \div 5^3$ if $x = 3$

(49) $x^{105} \div x^{97}$ if $x = -1$

(50) $(xy^2z^{15})^{12} \div \sqrt[3]{4a^{12}y^{15}z^{18}}$ if $x = 0$, $y = 2$, $z = 12$, $a = 30$

51 - 56 Explain the error and correct the solution in each of the following:

(51) A student simplified $(5ax^3)^2$ and got $25ax^6$.

(52) A student simplified $(\sqrt{x^4})^3$ and got x^{12}.

(53) A student simplified $(a^2)(b^3)$ and got $(ab)^5$.

(54) A student simplified $3^2 \cdot 3^4$ and got 9^6.

(55) A student simplified $18y^6 \div 9y^2$ and got $2y^3$.

(56) One generation back you had two ancestors (your parents). How many ancestors do you have 10 generations back? (Express your answer in exponential form and as an integer.)

(57) Suppose that you got a chain letter with a list of four names. The letter says to send your favorite recipe to the name at the top, cross that name out and add your name to the bottom of the list. Then you are to send a copy of the letter to four of your friends. If no one breaks the chain, how many recipes should you receive?

(58) Prove that $x^a \cdot x^b = x^{a+b}$ where a and b are positive integers and x is real.

(59) Prove that $\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$ where a is a positive integer and x and y are real.

(60) In the Laws of Exponents presented at the beginning of this section we stated that

$$x^a \div x^b = x^{a-b}, \quad a > b.$$

Consider the case where $a < b$. Remember that

$x^a = \underbrace{x \cdot x \ldots x}_{a \text{ factors}}$ and simply the expressions

$x^{10} \div x^{12}$, $x^{15} \div x^{20}$, $x^3 \div x^7$. In general what can you say about the simplified form of an expression $x^a \div x^b$ where $b > a$? (Hint: Write $\frac{x^{10}}{x^{12}}$.)

2.2 Characteristics of Powers

In the exercises of this section you will be asked to explore with the aid of your calculator some of the characteristics of various powers. This will often involve use of the $\boxed{y^x}$ key but sometimes you will wish to (or need to) revert to use of repeated multiplication.

EXAMPLE 1. Compute 5^7 exactly

Keystrokes

AOS $\boxed{5}$ $\boxed{y^x}$ $\boxed{7}$ $\boxed{=}$

RPN $\boxed{5}$ $\boxed{\text{ENT}}$ $\boxed{7}$ $\boxed{y^x}$

Answer: 78125

Some calculators will give close but not exact answers for such calculations. For example, here the answer might be 78124.8 or 78124.9 or 78125.1. In cases like this you can do one of several things:

(1) You can use common sense to adjust your answer. Here, for example, we know that the answer is an integer. The closest integer to the inexact result is 15625. Confirming evidence is the fact that any power of 5 has 5 in its units digit. ($5^2 = 25$, $5^3 = 125$, $5^4 = 625$, etc.)

(2) You can determine the highest power that does give an integer answer and complete the answer by using properties of powers. Here, for example, your calculator might give as the highest power of 5 for which the result is an integer

$$5^5 = 3125$$

The laws of exponents help you now

$$5^7 = 5^5 \cdot 5^2 = 3125 \cdot 25 = 78125$$

Sometimes you will be forced to find other means to compute exact answers as in the following example. Follow this example closely even if you have a calculator that displays more digits for the method is important.

EXAMPLE 2. Compute 5^{15} exactly.

This answer will not be displayed exactly on a six-digit display (neither will it on an eight-digit or even ten-digit display). Assume that the largest result we can compute is $5^8 = 390625$. There are several ways we could try to proceed:

(1) $5^{15} = 5^8 \cdot 5^7 = 390{,}625 \cdot 78{,}125$, or

(2) $5^{15} = 5^7 \cdot 5^7 \cdot 5 = 78{,}125 \cdot 78{,}125 \cdot 5$, or even

(3) $5^{15} = 5^8 \cdot 5^8 \div 5 = 390{,}625 \cdot 390{,}625 \div 5$.

Other approaches are also possible, but all lead to products that will not display. Clearly we must find a way to display the answer part at a time. We will use method (2) to do this.

We will compute 5^{14} and leave you to find a way to multiply this result by 5 (in exercise 23).

$5^{14} = 5^7 \cdot 5^7 = (5^7)^2$ by Exponent Laws (1) and (3)
We know $5^7 = 78125$ from EXAMPLE 1. We wish to square 78125.

Recall $(a + b)^2 = a^2 + 2ab + b^2$. Here we can let $a = 78{,}000$ and $b = 125$. We cannot calculate $(78{,}000)^2$ because it will overload, but we can calculate $(78)^2$ and affix the necessary six zeros.

Using $78^2 = 6{,}084$ and $125^2 = 15{,}625$ and $2 \cdot 78 \cdot 125 = 19500$ we have

$78{,}000^2 =$ 6,084,000,000

$2 \cdot 78{,}000 \cdot 125 =$ 19,500,000

$125^2 =$ 15,625

Adding gives $(78{,}125)^2 =$ 6,103,515,625

Notice that if we knew a way to represent $(a + b)^3$ we could have used $5^{15} = (5^5)^3 = (3{,}125)^3$ to get our answer. We will develop this formula and use it in exercise 24.

Exercises

Exponents increase (or sometimes decrease) the base rapidly. Try to accustom yourself to this rapid size change by guessing the smallest integral exponent needed to make the inequality true and then calculating the correct answer by direct trial.

1. $8^x > 10{,}000$
2. $2^y > 1{,}000{,}000$*
3. $3.1^p > 500{,}000$
4. $6^q > 40{,}000$
5. $(-4)^r > 250{,}000$
6. $(-4)^r < -250{,}000$

The following are important powers to remember. In each exercise n represents a positive integer.

7. $1^n =$
8. ** $0^n =$ (for $n \neq 0$)
9. $(-1)^{2n} =$
10. $(-1)^{2n+1} =$
11. a^{2n} is (positive, negative) for $a < 0$.
12. a^{2n+1} is (positive, negative) for $a < 0$.

---

*On a six-digit calculator y would be the first exponent to cause overflow.

**Your calculator may give you an Error message. In that case use your definition of exponents to determine the correct answer.

Now we wish to explore the effect of positive integral powers on bases in two ranges, $0 < b < 1$ and $1 < b$. We already know from exercise 7, the result of b^n for $b = 1$. Before doing any computing, guess which symbol, $=$, $>$, $<$, $\geq$ or $\leq$, should be placed between each pair of expressions. Then test your guess by calculating

13. b^n __ b^{n+1} for $b > 1$. (Include in testing, b values like 1.01, 1.001.)

14. b^n ___ b^{n+1} for $0 < b < 1$. (Include b values like 0.999.)

Calculate exact digits for:

15. 23^4 16. 23^4 another way

17. 23^4 by a third method 18. $(.023)^4$

19. Choose a, b or c depending on your calculator:
    (a) for six-digit display 6^8
    (b) for eight-digit display 6^{11}
    (c) for ten-digit display 6^{13}

20. (a) for six-digit display 12^6
    (b) for eight-digit display 12^8
    (c) for ten-digit display 12^{10}

21. 2^{17} 22. 2^{34} (Hint: use 2^{17}.)

23. 5^{15} (Hint: see text Example 2 for 5^{14}.)

EXPLORATION EXERCISE

24. We seek 5^{15} by cubing 5^5. Find $(a + b)^3$ by multiplying $(a^2 + 2ab + b^2)(a + b)$. Check that $5^5 = 3125$. Use the identity you have developed for $(a + b)^3$ with $a = 3100$ and $b = 25$ to calculate 5^{15}. Your reasoning will be of the form:

$$5^{15} = (5^5)^3 = (3125)^3 = (3100 + 25)^3 = 3100^3 + 3 \cdot 3100^2 \cdot 25 + \ldots\ldots$$

2.3 Scientific Notation (for Numbers Greater than One)

The distance from the earth to the sun is about 93,000,000 miles (93 million miles). As science expands its frontiers such large numbers turn up more and more often. Note that they do not apply only to astronomical distances. For example in chemistry, Avagadro's number, the number of molecules in a given small unit of volume, is

602,000,000,000,000,000,000,000

Scientists (who are as lazy as the rest of us when it comes to writing out things like this) have developed several short notations for such numbers. Here we will consider only one: Scientific Notation. Others will be considered in the exercises. In scientific notation, for example, Avagadro's number is 6.02×10^{23} and the distance from the earth to the sun is

$$9.3 \times 10^{7} \text{ miles.}$$

The notation is based on the special role of the number ten in our decimal number system. Some familiar properties of ten are:

(1) Multiplying by ten moves the decimal point one place to the right.

$$57 \times 10 = 570 \qquad 3.2 \times 10 = 32 \qquad 5.834 \times 100 = 583.4$$

(2) Dividing by ten moves the decimal point one place to the left.

$$57 \div 10 = 5.7 \qquad 3.2 \div 10 = .32 \qquad 5.834 \div 100 = .05834$$

(3) 10^{n} (for whole number n) is the same as $1\underbrace{0\ldots00}_{n \text{ zeros}}$

$$10^{3} = 1000 \qquad 10^{7} = 10\ 000\ 000 \qquad 10^{2} = 100$$

Scientific Notation makes use of these properties of 10.

To represent a number in scientific notation the following form is used. The number is represented:

$$\boxed{\begin{array}{c}\text{a number}\\ \text{between}\\ \text{one and ten}^*\end{array}} \times 10^n$$

(In this section n will be a whole number)

Examples: 3.2×10^2 $\quad 4.835 \times 10^{61}$ $\quad 8.1 \times 10^{15}$ $\quad 1.0 \times 10^5$

All numbers ten or larger may be represented in this form. Now what number in standard notation do these numbers represent? You can use the following simple rule to make the change:

$$6.45 \times 10^n = 6\underrightarrow{.45}$$

Move the decimal point past n digits
(annexing zeros to do so if necessary)

Examples:

$3.2 \times 10^2 = 320$ (decimal point moved two places)

$1.0 \times 10^5 = 100\ 000$

$8.1 \times 10^{15} = 8\ 100\ 000\ 000\ 000\ 000$

$3.8295 \times 10^3 = 3829.5$

When you see numbers in scientific notation you will wish to think of them in this way.

Now we are left with the question, how do we change numbers to scientific notation? The procedure is not difficult and involves only thinking of multiplying and dividing a number by the same power of ten.

To change 93,000,000 to scientific notation, for example,

$$93{,}000{,}000 = \frac{93{,}000{,}000}{10{,}000{,}000} \times 10{,}000{,}000 = 9.3 \times 10^7$$

---

*More accurately $1 \leq n < 10$.

Notice how we divide and multiply by the same number thus not changing the value of the number but only the form of the representation.

A rule for doing this is:

(1) Shift the decimal point n places to the left to create a new number between one and ten.

(2) Write the new number $\times 10^n$.

Examples:

$93,000,000 = 9.3 \times 10^7$

shift 7 places

$384.75 = 3.8475 \times 10^2$

shift 2 places

On a calculator scientific notation is displayed for large numbers beyond the regular display capability of the machine. For example, an eight digit display would not be able to handle a number over 100,000,000. For example, 384,000,000 would be displayed as:

3.84 08

to represent

3.84×10^8

Some computers would display this as 3.84 E08 (E for exponent of ten).

Although individual calculators vary, the following keys and procedures are often used. You will wish to check them against your own equipment.

$\boxed{\text{EE}}$ or $\boxed{\text{EEX}}$ This key is used to enter a power of ten

Example: To enter 3.84×10^5, key

$\boxed{3}$ $\boxed{\cdot}$ $\boxed{8}$ $\boxed{4}$ $\boxed{\text{EE}}$ $\boxed{0}$ $\boxed{5}$ display: 3.84 05

$\boxed{\text{SCI}}$ $\boxed{n}$ Numbers are represented in scientific notation rounded to n digits.

Example: With the number 38652000 in the display

$\boxed{\text{SCI}}$ $\boxed{3}$ would change the display to 3.865 07

Exercise set 2.3

1-10. Express each of the following in ordinary decimal notation.

(1) 8.37×10^6	(2) 5.63×10^3
(3) 2.9×10^4	(4) 2.847×10^1
(5) 6.273×10^2	(6) 3.3338×10^3
(7) 3.15×10^7	(8) 9.537×10^2

(9) the earth's mass is 1.32×10^{25} pounds.

(10) the velocity of light is 3×10^{10} centimeters per second.

(11) the age of the earth's crust is 5×10^9 years.

12-21. Express each of the following in scientific notation.

(12) 69,530	(13) 834,732
(14) 146	(15) 100,000
(16) 147,324	(17) 532.8
(18) 184.35	(19) 2376.4

(20) the approximate distance between our solar system and its nearest known star, Alpha Centuri, is 25,000,000,000,000 miles.

(21) A light year is the distance light travels in a year. A light year is approximately 5,878,000,000,000 miles.

(22) The number of atoms in a gram of hydrogen is approximately 600,000,000,000,000,000,000,000.

23-30 Simplify each of the following:

(a) by using your calculator and scientific notation

and (b) prove your result by writing the numbers out in decimal notation

example

(a) $3.84 \times 10^5 + 2.95 \times 10^3$

$\boxed{3}\ \boxed{.}\ \boxed{8}\ \boxed{4}\ \boxed{EE}\quad \boxed{5}\ \boxed{+}\ \boxed{2}\ \boxed{.}\ \boxed{9}\ \boxed{5}\ \boxed{EE}\ \boxed{3}\ \boxed{=}$

display 3.8695 05

(b) $3.84 \times 10^5 = 384000$

$+ 2.95 \times 10^3 = + \underline{\ 2950}$

$386950 = 3.8695 \times 10^5$

(23) $6.24 \times 10^5 + 3.21 \times 10^4$

(24) $\sqrt{1.44 \times 10^8}$

(25) $(5.6 \times 10^4)(2.12 \times 10^2)$

(26) $(2.5 \times 10^5)^2$

(27) $(4 \times 10^3) - (2.5 \times 10^2)$

(28) $(5 \times 10^6) \div (2.5 \times 10^2)$

(29) $\dfrac{(9.3 \times 10^6)(5 \times 10^2)}{(1.25 \times 10^4)}$

(30) $\dfrac{(4 \times 10^2)^3\ (8 \times 10^4)^2}{(6.25 \times 10^7)}$

2.4 New Exponents: Rational Powers

Until now we have restricted our use of exponents to the natural numbers. This was necessary because our definition related to counting:

$$\underbrace{a \cdot a \cdot a \cdot a \cdot a}_{\text{5 factors}} = a^5$$

We now wish to extend our definition to include powers like 0, -5, 1.8, $-\frac{1}{5}$, etc. To do this we must extend our definition. In doing this we wish to retain the simple rules for exponents of section 2.1. For that reason we merely apply those rules to situations where the counting method won't work and add new definitions when necessary.

EXAMPLES:

1. $\frac{a^b}{a^b} = 1$ and

$\frac{a^b}{a^b} = a^{b-b} = a^0$ Extending the division rule (2) of section 2.1 without justification.

Thus we want $a^0 = 1$ so we make that new definition:

$$\boxed{\text{DEFINITION:} \quad a^0 = 1 \quad \text{for all } a \neq 0}$$

EXAMPLES:

$5^0 = 1 \quad 283^0 = 1 \quad -49.38^0 = 1 \quad 10^0 = 1$

2. Similarly consider

$$\frac{a^5}{a^7} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{a^2} \quad \text{and}$$

$\frac{a^5}{a^7} = a^{-2}$ Extending the division rule (2) of section 2.1 without justification.

Thus we want $a^{-2} = \frac{1}{a^2}$. To ensure this we again add a new definition

DEFINITION: $a^{-b} = \frac{1}{a^b}$ for all $a \neq 0$

Examples: $5^{-3} = \frac{1}{5^3} = \frac{1}{125}$ $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = .01$

$.01^{-2} = (\frac{1}{100})^{-2} = 10000$ $(\frac{2}{3})^{-2} = (\frac{3}{2})^2 = \frac{9}{4}$

Note that the negative sign in the exponent signals RECIPROCAL.

3. Finally

$\sqrt[3]{a^5} = a^{\frac{5}{3}}$ Applying the power rule (4) of section 2.1 without justification.

This leads us to our final definition to allow extension of our rules:

DEFINITION: $a^{\frac{p}{q}} = \sqrt[q]{a^p}$ for $a > 0$

Since we have carefully extended our system by applying the rules of exponents from section 2.1 ALL OF THOSE RULES STILL APPLY!

Examples: $a^{1.6} \cdot a^{.4} = a^2$ by rule

$$\frac{a^2}{a^{-5}} = a^7 \qquad \text{by rule}$$

$$\sqrt[3]{a} \cdot \sqrt{a} = a^{\frac{1}{3}} \cdot a^{\frac{1}{2}} = a^{\frac{5}{6}} \quad (\text{since } \tfrac{1}{3} + \tfrac{1}{2} = \tfrac{5}{6})$$

$$= \sqrt[6]{a^5}$$

In the exercises you will have a chance to apply these extended rules and to check them against calculator computations.

Exercises

(1 - 8) Write each of the following with a radical sign

(1) $y^{\frac{1}{3}}$ (2) $2x^{\frac{1}{2}}$

(3) $-5^{\frac{1}{4}}$ (4) $e^{\frac{x}{y}}$

(5) $ab^{-\frac{2}{3}}$ (6) $(2a)^{-\frac{1}{6}}$

(7) $\left(\frac{x^{o}}{y}\right)^{\frac{1}{2}}$ (8) $3a^{-\frac{3}{5}}$

(9 - 22) Find the value of each of the following

(9) $36^{-\frac{1}{2}}$ (10) $16^{-\frac{3}{4}}$

(11) $27^{\frac{1}{3}}$ (12) $(\frac{2}{3})^{-3}$

(13) $3(49)^{\frac{1}{2}}$ (14) $2(3)^{-4}$

(15) $(\frac{1}{81})^{\frac{1}{2}}$

(16) $\frac{1}{4}(2)^{-3}$

(17) $(\frac{25}{36})^{-\frac{1}{2}}$

(18) $\sqrt{10^{-6}}$

(19) $-8^{\frac{1}{3}}$

(20) $(.008)^{\frac{1}{3}}$

(21) $4\left(\frac{1}{8}\right)^{\frac{2}{3}}$

(22) $\left(\frac{4}{9}\right)^{\frac{5}{2}}$

(23 - 34) Simplify each of the following

(23) $a^{\frac{4}{3}} \cdot a^{-\frac{1}{3}}$

(24) $\left(z^{-\frac{2}{5}}\right)^{-\frac{1}{2}}$

(25) $y^3 \div y^{-5}$

(26) $\sqrt[3]{x^{4/3}}$

(27) $a^{\frac{3}{4}} \div a^{\frac{1}{2}}$

(28) $x^5y^{-3} \cdot x^{-2}y^{-4}$

(29) $d^{-\frac{1}{4}} \cdot d^{-\frac{1}{2}}$

(30) $\left(x^{-\frac{3}{4}}\right)^{a}$

(31) $8^{\frac{5}{3}} \cdot 3^{-1}$

(32) $8^{\frac{1}{3}} + 2^{-1} - 3^0$

(33) $\left(\frac{x^{3a}}{y^{6b}}\right)^{\frac{2}{3}}$

(34) $27^{-\frac{2}{3}} - 3^{-1}\ (25)^{\frac{1}{2}}$

Exercises 9 - 22 can be evaluated by calculator computation. The answers however will be expressed as decimals rather than as rational expressions.

Example:

$$36^{-\frac{1}{2}}.$$

keystroke sequence for RPN calculators

$\boxed{3}$ $\boxed{6}$ $\boxed{\text{ENTER}}$ $\boxed{1}$ $\boxed{\text{ENTER}}$ $\boxed{2}$ $\boxed{\div}$ $\boxed{\text{CHS}}$ $\boxed{f}$ $\boxed{y^x}$

or

$\boxed{3}$ $\boxed{6}$ $\boxed{\text{ENTER}}$ $\boxed{.5}$ $\boxed{\text{CHS}}$ $\boxed{f}$ $\boxed{y^x}$

keystroke sequence for algebraic calculators

$\boxed{3}\ \boxed{6}\ \boxed{y^x}\ \boxed{(}\ \boxed{1}\ \boxed{\div}\ \boxed{2}\ \boxed{\text{CHS}}\ \boxed{)}\ \boxed{=}$

display 0.1667

your answer to #9 was $\frac{1}{6}$. Recall that $\frac{1}{6} = .166\dot{6}$

(35 - 48) Go back and do exercises 9 - 22 by using your calculator. Verify that your answers are equivalent.

Often when doing calculations involving exponents on your calculator it is convenient to represent exponents in decimal rather than rational form.

Example $81^{\frac{3}{4}} = 81^{.75} = 27$

$64^{\frac{2}{3}} = 64^{.66\dot{6}} = 16$

$64^{\frac{3}{2}} = 64^{1.5} = 512$

Furthermore a calculation such as $\sqrt[3]{5} \cdot \sqrt[2]{5}$ can be evaluated in 2 ways.

method (a) $\sqrt[3]{5} \cdot \sqrt[2]{5} = 1.7100 \cdot 2.2361 = 3.8236$

method (b) $\sqrt[3]{5} \cdot \sqrt[2]{5} = 5^{\frac{1}{3}} \cdot 5^{\frac{1}{2}} = 5^{\frac{5}{6}} = 3.8236$

(49 - 60) Perform each of the following calculations on your calculator in at least two ways. Fix your calculator to four decimal places.

(49) $\sqrt[5]{32^3}$

(50) $7^{-\frac{1}{4}} \div 2^{-\frac{1}{4}}$

(51) $\sqrt{8} \cdot \sqrt[4]{8}$

(52) $\left(\sqrt[4]{81^3}\right)^{-2}$

(53) $\frac{1}{100}^{-\frac{1}{2}} \div \sqrt{100}$

(54) $\sqrt{0.23} \cdot \sqrt[3]{0.23}$

(55) $\left(\sqrt[5]{3^2}\right)^4$

(56) $(0.015625)^{\frac{1}{3}} \div \left(\frac{1}{4}\right)^3$

(57) $\sqrt[3]{27^2} \cdot \sqrt[2]{27^3}$

(58) $\sqrt[4]{2401} \div \left(\frac{1}{49}\right)^{-\frac{1}{2}}$

(59) $\left(\sqrt[3]{4^2}\right)^{-\frac{3}{2}}$

(60) $\sqrt{2} \cdot \sqrt[4]{4}$

(61) Why is 1 your answer to 58?

(62) Why is 2 your answer to 60?

2.5 Scientific Notation (for Numbers near Zero)

In this section we extend the methods of Section 2.3 to small numbers. To do this we use the tools provided in Section 2.4.

Examine the following examples in which small numbers are rewritten in scientific notation in order to determine the patterns:

EXAMPLES:

$$.23 = \frac{23}{100} = \frac{23}{10^2}{}^{*} = 23 \times 10^{-2} = 2.3 \times 10 \times 10^{-2} = 2.3 \times 10^{-1}$$

$$.0007 = \frac{7}{10000} = \frac{7}{10^4} \overset{*}{=} 7 \times 10^{-4}$$

$$.00372 = \frac{372}{100000} = \frac{372}{10^5} \overset{*}{=} 372 \times 10^{-5} = 3.72 \times 10^2 \times 10^{-5} = 3.72 \times 10^{-3}$$

Notice first how we used (at *) the fact that

$$\frac{1}{10^n} = 10^{-n}.$$

Then focus only on the given number in its two forms

standard notation	scientific notation
.23	2.3×10^{-1}
.0007	7×10^{-4}
.00372	3.72×10^{-3}

Do you notice how the number of places the decimal point is moved relates to the power of ten?

$$\underbrace{.23}_{\text{shift 1 place}} = 2.3^{-1}$$

$$\underset{\longrightarrow}{.003}72 = 3.72 \times 10^{-3}$$

shift 3 places

Of course this is the same relationship between decimal point shift and power of ten that we found in Section 2.3. Therefore we can say in general

DECIMAL POINT SHIFT ⟷ POWER OF TEN

The power of ten is negative for small numbers ($-1 < n < 1, n \neq 0$) and positive for large numbers ($n > 10$ or $n < -10$).

This leaves only one power of ten to explore, 10^0. Clearly, if the rule applies here as well, this would mean a decimal shift of zero places.

EXAMPLE

$$2.78 = 2.78 \times 10^0$$

Since $10^0 = 1$, this is reasonable. Thus numbers in the range $1 \leq n < 10$ (and $-10 < n \leq -1$) are represented with no decimal point shift and 10^0.

Your calculator will carry out complex calculations with numbers in scientific notation, but you should always be prepared to explain what is happening. Otherwise you will not be alert to errors. For example suppose we wish to calculate the reciprocal of

$$4 \times 10^5$$

Pressing the appropriate keys gives

$$2.5 \times 10^{-6}$$

Why? You must examine the process carefully

$$\frac{1}{4 \times 10^5} = \frac{1}{4} \times \frac{1}{10^5} = .25 \times 10^{-5} = 2.5 \times 10^{-1} \times 10^{-5} = 2.5 \times 10^{-6}$$

Accuracy

One of the values of scientific notation that we do not explore in detail here is its direct expression of the accuracy (or inaccuracy) of the number expressed. Avogadro's Number, for example, is accurately represented as

$$6.02 \times 10^{23}$$

because this is the value to three digit accuracy. The value given at the beginning of section 2.3 is approximate with no indication of how large the error is. This will be explored briefly in the exercises.

Exercises

1 - 8 Express each of the following numbers in scientific notation:

(1) .000000412 (2) .002578
(3) .137 (4) 1247503
(5) .02372 (6) 2.301
(7) .10026 (8) .000000000785
(9) Acceleration due to gravity is 980.665 cm/sec^2.
(10) One kilowatt-hour is 864,000 calories.

11 - 20 Express each of the following numbers in standard notation.

(11) 1.47×10^{-4} (12) 2.563×10^{6}
(13) 5.7×10^{7} (14) 1.03×10^{-5}
(15) 3×10^{-6} (16) 6.82×10^{-8}
(17) 2.69×10^{10} (18) 4.57×10^{-2}
(19) the mass of an electron at rest is 9.1066×10^{-28} gm.
(20) the speed of light in a vacuum is 2.99776×10^{10} cm/sec.

(21) How is zero represented in scientific notation in your calculator?

(22) How else could zero be represented in your calculator?

Numbers used by scientists are generally measurements. The accuracy of any measurement is always limited and hence the number expressed should be written with the number of digits that properly expresses the accuracy of the measurement. These figures and only these are significant. When computations are made with numbers obtained experimentally, the number of digits retained in the result is determined by the number of significant figures in the original data.

Examples:

length of a page = 22.7 cm (3 significant figures)
thickness of page = 0.011 cm (2 significant figures)
distance to the sun = 93,000,000 mi. (2 significant figures)
speed of light = 299,780 km/sec (5 significant figures)

If each of these is written in scientific notation there is no doubt as to the number of significant figures for only the significant figures are retained

2.27×10^{1} cm
1.1×10^{-2} cm
9.3×10^{7} cm
2.9978×10^{5} km/sec

Scientific notation implies the precision of the measurement expressed. The number 2.27 is correct to the nearest hundredth because the last significant digit is in the hundredths place. Thus it is implied that the length of a page is within 22.65 cm and 22.74 cm. The distance to the sun is correct to the nearest 1 million. Thus it is implied that the distance to the sun is between 92,500,000 mi. and 93,400,000 mi.

Zero can be a significant digit. 2.3×10^{2} has a different precision than 2.30×10^{2} because 2.3×10^{2} is correct to the nearest ten and has a range from 225 to 234 while 2.30×10^{2} is correct to the nearest unit and has a range from 229.5 to 230.4 Notice that standard notation cannot make this distinction.

23 - 27 For each of the following measurements determine

a) the number of significant digits of the measurement

b) the precision of the measurement (correct to the nearest ___)

c) the range of error of the measurement

(23) 1.324×10^4 (24) 2.7×10^{-2}

(25) 6.1×10^{-2} (26) 6.10×10^{-2}

(27) 3.267×10^5

Engineering notation is a modified form of scientific notation. All numbers are shown with exponents of 10 that are multiples of 3.

$$\boxed{\begin{array}{c}\text{a number between}\\ \text{1 and 1000}^*\end{array}} \times 10^m \quad (m \text{ a multiple of } 3)$$

Examples:

standard notation	engineering notation	scientific notation
5280	5.28×10^3	5.28×10^3
528	528×10^0	5.28×10^2
.0528	52.8×10^{-3}	5.28×10^{-2}

This form of notation is particularly useful in scientific and engineering calculations where units of measure are often specified in multiples of three. Some auxiliary metric prefixes further streamline the system.

prefix**	definition	prefix	definition
tera	1×10^{12}		
giga	1×10^9	milli	1×10^{-3}
mega	1×10^6	micro	1×10^{-6}
kilo	1×10^3	nano	1×10^{-9}
		pico	1×10^{-12}

---

*more accurately $1 \le n < 1000$.

**Two other metric prefixes commonly used are deci (1×10^{-1}) and centi (1×10^{-2}) but these do not fit engineering notation.

Using this system a surveyor who is working with a distance of 1,432,000 meters can express this distance as 1.432 megameters. Similarly the diameter of a blood cell is 0.0000075 meters or 7.5 micrometers (generally shortened to microns).

28 - 35 Represent each of the following in engineering notation and check your answer on your calculator. Fix your calculator to an engineering format for 5 decimal places.

(28) .001234
(29) .04732
(30) .00000001
(31) 1234.56
(32) 1.237 kilometers to meters
(33) 8.37 nanoseconds to seconds
(34) 6.32 megatons to tons
(35) 2.04 milliliters to liters

2.6 Solving Equations with Exponents

You may now use the knowledge you have gained of exponents to solve kinds of equations different from those you have studied before.

EXAMPLE Solve $x^{-2} = 16$

Since $x^{-2} = \frac{1}{x^2}$, the equation is the same as

$$\frac{1}{x^2} = 16$$

$$1 = 16x^2$$

$$\frac{1}{16} = x^2$$

$$\pm \frac{1}{4} = x$$

EXAMPLE Solve $x^{4/3} - 16 = 0.$

First, transform the equation to the form $x^{\pm\frac{p}{q}} = k.$

$$x^{4/3} = 16$$

Now you may apply two methods

(1) $x^{4/3} = \sqrt[3]{x^2}$ so $\sqrt[3]{x^2} = 16$

Taking the square root of each member

$$\sqrt[3]{x} = \pm 4$$

Cubing

$$x = \pm 64$$

(2) Cube each member of $\sqrt[3]{x^2} = 16$

$$x^2 = 4096$$

Taking the square root of each member

$$x = \pm 64$$

Note that method (1), taking roots first, keeps numbers small, and is preferable for that reason.

There is another approach to such examples that we recommend. Recall the power rule (3 on page 2.1 - 2):

$$(x^a)^b = x^{ab}$$

In the examples we sought to convert the forms

$$x^{-2} = 16 \qquad \text{and} \qquad x^{4/3} = 16$$

each to the form

$$x = k \quad (\text{or alternately } x^1 = k)$$

To do this we raise each member of the equation to the same power:

$$(x^{-2})^? = 16^? \qquad \left(x^{4/3}\right)^? = 16^?$$

in each case selecting the power that will lead to x^1:

$$(x^{-2})^{-\frac{1}{2}} = \pm 16^{-\frac{1}{2}} \qquad (x^{\frac{2}{3}})^{\frac{3}{2}} = \pm 16^{\frac{3}{2}}$$

$$x^1 = \frac{1}{\pm\sqrt{16}} = \pm\frac{1}{4}\,{}^{*} \qquad x^1 = \pm\sqrt{16^3} = \pm 4^3 = \pm 64^{*}$$

Another kind of equation you should now be able to solve has the variable as exponent: this is called an exponential equation.

EXAMPLE $8^{x+2} = 16^{2x-1}$

To solve such an equation we seek the same base for each member: 2 is such a base. Substituting 2^3 for 8, 2^4 for 16:

$$(2^3)^{x+2} = (2^4)^{2x-1}$$
$$2^{3x+6} = 2^{8x-4}$$

Now since the bases are the same, the exponents must be equal

$$3x + 6 = 8x - 4$$
$$10 = 5x$$
$$2 = x$$

EXAMPLE $(.5)^x = 4$

(Don't forget that $.5 = \frac{1}{2}$.) Two is a convenient base:

---

[*] Caution must be exercised in dealing with fractional exponents. $16^{\frac{1}{2}} = \sqrt{16} = 4$, in each case the radical or exponent indicating the principal (positive) root. In solving an equation, however, you do not wish to lose negative roots. For that reason, when you apply square root (or a power with even denominator) you should always affix ± to the resulting expression!

$$(2^{-1})^x = 2^2$$

$$2^{-x} = 2^2$$

$$-x = 2$$

$$x = -2$$

Exercises:

1 - 10 Solve and check each of the following:

(1) $x^{1/2} = 7$

(2) $x^{1/3} - 2 = 3$

(3) $5y^{1/4} = 10$

(4) $a^{4/3} = 8$

(5) $y^{-2} = 25$

(6) $16c^{-4} + 5 = 6$

(7) $x^{-1/3} = \frac{1}{7}$

(8) $(9w)^{-1/2} = 1$

(9) $54n^{-2/3} = 6$

(10) $80m^{-4/3} = 5$

11 - 30 Solve and check each of the following:

(11) $2^x = 16$

(12) $3^x - 1 = 80$

(13) $7^x = 1$

(14) $5^x = \frac{1}{125}$

(15) $25^{2y} = 5$

(16) $2^{x+3} = 256$

(17) $256 = 4^{3t-2}$

(18) $4 \cdot 2^4 = 8^a$

(19) $5^{2y+3} = \frac{1}{5}$

(20) $7^{3x+1} = 49^{x-1}$

(21) $8^{x-2} = 64^{2x+2}$

(22) $9^{x-2} = 3^{x+1}$

(23) $27^{x+2} = \left(\frac{1}{3}\right)^{2-x}$

(24) $100^{x+2} = 10000^{x-1}$

(25) $\left(5^{4x-3}\right)^{-2} = 125^{-x-8}$

(26) $\left(3^3\right)^{2x-6} = \left(3^{-2}\right)^{3-x}$

(27) $3^{3-x}\left(9^{2x-1}\right) = 81$

(28) $9^x\left(3^{4x}\right) = \frac{1}{27}$

(29) $(6^{2x})(36) = 216^{x-1}$

(30) $3^{x+3} + 3^{x+2} = 12$

31 - 35 Decide whether each of the following is true or false:

(31) If $x^{1/2} = 25$ then $x = 5$

(32) If $1^x = 1^4$ then $x = 4$

(33) If $\left(\frac{1}{2}\right)^x = 4$ then $x = -2$

(34) If $2^x = +1$ then $x = 0$

(35) If $5^{-x} = 25$ then $x = \frac{1}{2}$

36 - 39 All of the preceding exercises have been contrived to "come out even", that is, to give values that are integers or at least reasonable fractions. You can use your calculator to determine approximate answers to other exercises

EXAMPLE Solve $5^x = 19$ to thousandths

Since $5^1 = 5$ and $5^2 = 25$ we know $1 < x < 2$

Using the calculator with the keystroke sequence

ALGEBRAIC $\boxed{5}$ $\boxed{y^x}$ trial power $\boxed{=}$

RPN $\boxed{5}$ $\boxed{\text{ENT}}$ trial power $\boxed{y^x}$

gives the following

x	5^x	
1.5	11.1803	
1.8	18.12	
1.9	21.28	$1.8 < x < 1.9$
1.85	19.64	
1.83	19.02	
1.82	18.71	$1.82 < x < 1.83$
1.825	18.86	
1.828	18.95	
1.829	18.99	$1.829 < x < 1.830$
1.8295	19.00053	$x = 1.829$

Find x to the nearest hundredth

(36) $3^x = 100$

(37) $10^x = 50$

(38) $10^x = 500$

(39) $10^x = 5$

2.7 Using Exponents to Calculate

We will introduce this new concept indirectly. First we construct a table of powers of 4.

n	0	1	2	3	4	5	6	7	8
4^n	1	4	16	64	256	1024	4096	16384	65536

Now we notice an important short-cut for multiplying numbers in the lower row of the table.

EXAMPLE $64 \times 256 = ?$

Add the numbers about 64 and 256 (3 + 4) and look below the sum (7) for the answer 16384.

Try this shortcut yourself for other products like 16×4096, 64×64, and 256×256.

Of course the reason for the short-cut should be clear to you. You have performed the multiplication in the following way:

64×256	
$4^3 \times 4^4$	from the table
4^7	by the rules for exponents.
16384	from the table.

We can extend the table slightly by using some other exponent rules. For example, $4^{1.5} = 4^{6/4} = 2^3 = 8$, etc.

n	4^n
0	1
.5	2
1	4
1.5	8
2	16
2.5	32
3	64
3.5	128
4	256
4.5	512

n	4^n
5	1024
5.5	2048
6	4096
6.5	8192
7	16384
7.5	32768
8	65536
8.5	131,072
9	262,144
9.5	524,288
10	1,048,576

With the new table you can perform still more multiplications using the same short-cut

EXAMPLE

$$\begin{array}{cccccl} 128 & & 2048 & & & \\ \updownarrow & & \updownarrow & & & \\ 3.5 & + & 5.5 & = & 9 & \text{exponents of 4} \\ & & & & \updownarrow & \\ & & & & 262144. & \end{array}$$

Try other products like 32 × 1024 and 32768 × 32.

It would seem that this method could be extended to other calculations if we could represent other powers of four, like $4^{2.7}$ for example. In fact exactly this is true. We will show some examples of this in the following table. You may check our values with your calculator (using the $\boxed{y^x}$ key).

n	4^n
0	1
.1	1.1487
.2	1.3195
.3	1.5157
.4	1.7411
.5	2.0000
.6	2.2974
.7	2.6390
.8	3.0314

n	4^n
.9	3.4822
1.0	4.0000
1.1	4.5948
1.2	5.2780
1.3	6.0629
1.4	6.9644
1.5	8.0000

Now we can calculate other products (approximately) by the same shortcut:

$$\begin{array}{lcccccl} \text{EXAMPLE} & 1.7411 & \times & 2.6390 & & & \\ & \updownarrow & & \updownarrow & & & \\ & .4 & + & .7 & = & 1.1 & \text{exponents of 4} \\ & & & & & \updownarrow & \\ & & & & & 4.5948 & \end{array}$$

Check to see that other products may be calculated by the same shortcut, for example 1.1487 × 2.2974 = 2.6390 and 1.5157 × 2.6390 ≐ 4. (Notice how in the last example the answer is not exact.)

Exercises:

1 - 8 By using your calculator, extend the table on page 2.7 - 2 by tenths to 3.0. Use the two tables to calculate

(1) 3.0314 × 18.3792

(2) 1.3195 × 32

(3) 9.1896 × 5.278

(4) 6.0629 × 6.0629

(5) 6.9644 × 9.1896

(6) 13.9288 × 3.4822

(7) 1.1487 × 42.2243

(8) 4 × 21.1121

9 - 17 There are problems with our procedure. For one thing we have the "nice" numbers in the left hand column (n) and only a few reasonable numbers (the exact powers of 2) in the right column. We are faced with the problem: How can we multiply 5 × 19 by this means? We will explore this problem in the following exercises.

9 - 11 We seek n in $4^n = 5$

(9) Between what two integers is n?

(10) From your tables locate n between values to tenths.

(11) Use your calculator to find n (by trial and error) to hundredths.

12 We seek n in $4^n = 19$

(12) Locate n between two integers.

(13) Locate n between tenths from your tables.

(14) Use your calculator to find n to hundredths.

(15) Enter your answers from exercises 11 and 14 in the following tables.

n	4^n
	5
	19

(16) Use the "short-cut" to find the product 5 × 19. (You know the product but check it by adding values of n and then raising 4 to that power with your calculator.)

(17) Why do you think your answer is not exactly 95?

18 - 26 We have used the tables to multiply. They can also be used to divide.

(18) $4^5 \div 4^3 = 4^x$ What is x?

(19) To divide numbers in the 4^n column, ______numbers in the n column.
Use this short-cut and your tables to calculate:

(20) $18.3792 \div 3.0314$

(21) $13.9288 \div 3.4822$

(22) $42.2243 \div 6.0629$

(23) $21.1121 \div 2.6390$

(24) $48.5029 \div 6.9644$

(25) $55.7152 \div 4.5948$

(26) Use your answer to exercise 15 to calculate $19 \div 5$ by this means.

27 - 30 Make a table of powers of 3 by tenths from 0 to 1. Use it to calculate

(27) 1.3904×1.5518 (28) 1.2457×2.1577

(29) 1.11612×2.6879 (30) 1.3904×1.3904

2.8 Powers of Ten

In section 2.7 we constructed tables relating powers of 4 (and 3 in the exercises) to numbers and used those tables to calculate. This method is widely used in science for many more important purposes in addition to calculation, but instead of the bases 3 or 4 the following are more often used:

e ($\doteq 2.7^+$)
2
10

While the first two bases are very important, in this section we will study only base 10. Many of the ideas we develop about this base will apply to the other bases as well.

Preliminary Exercises:

Copy and complete the following tables using your calculator when necessary:

1.

n	10^n
0	
1	
2	
3	
4	

2.

n	10^n
0	
-1	
-2	
-3	
-4	

3.

n	10^n (to 4 decimal digits)
0	
.1	
.2	
.3	
.4	
.5	
.6	
.7	
.8	
.9	
1.0	

4.

n	10^n
1.1	
2.1	
3.1	
4.1	
.1	

5.

n	10^n
.7	
1.7	
2.7	
3.7	

Now, of course, you can calculate the same kinds of exercises that you did with the 3's and 4's power tables in section 2.7.

EXAMPLE. Use the table of exercise 3 to calculate

$$\begin{tikzcd}
1.5849 \arrow[d, leftrightarrow] & \times & 3.1623 \arrow[d, leftrightarrow] & & \\
.2 & + & .5 & = & .7 \arrow[d, leftrightarrow] \\
 & & & & 5.0119
\end{tikzcd}$$

(exponents of 10)

Here you are applying basic rules of exponents.

$$\begin{array}{rrl} & 1.5849 = & 10^{.2} \\ \times & 3.1623 = & \underline{10^{.5}} \\ & & 10^{.7} = 5.0119 \end{array}$$

$$\boxed{10^{.2} \cdot 10^{.5} = 10^{.7}}$$

Study the following examples carefully.

EXAMPLE:

$$\begin{tikzcd}
7.9433 \arrow[d, leftrightarrow] & \div & 2.5119 \arrow[d, leftrightarrow] & & \\
.9 & - & .4 & = & .5 \arrow[d, leftrightarrow] \\
 & & & & 3.1623
\end{tikzcd}$$

$$\boxed{10^{.9} \div 10^{.4} = 10^{.5}}$$

EXAMPLE:

$$\begin{tikzcd}
(1.5849)^3 \arrow[d, leftrightarrow] & & & & \\
.2 & \times 3 & = & .6 \arrow[d, leftrightarrow] \\
 & & & 3.9811
\end{tikzcd}$$

$$\boxed{(10^{.2})^3 = 10^{.6}}$$

EXAMPLE:

$$\begin{tikzcd}
\sqrt[4]{6.310} \arrow[d, leftrightarrow] & & & & \\
.8 & \times & \frac{1}{4} & = & .2 \arrow[d, leftrightarrow] \\
 & & & & 1.5849
\end{tikzcd}$$

$$\boxed{(10^{.8})^{1/4} = 10^{.2}}$$

You will explore some additional properties of powers of ten in the exercises.

Exercises:

(1) Extend the table of preliminary exercise 3 to 2 (by tenths).

(2-10) Use the tables of preliminary exercise 3 and exercise 1 to calculate the following by short-cut.

(2) 1.2589×3.9811 (3) 1.9953×7.9433

(4) $10 \div 2.5119$ (5) $6.3096 \times 2.5119 \div 5.0119$

(6) $\sqrt{25.1189}$ (7) $(2.5119)^5$

(8) $(7.9433)^{3/4}$ (9) $\sqrt[3]{31.6228}$

(10) $\dfrac{3.9811 \times 5.0119^2}{31.6228}$

(11) Compare your table entries for $n = .1$ and $n = 1.1$. How do they relate?

(12) Generalize exercise 11.

(13) What would you expect to be the table entry for $n = 2.1$? 3.1?

(14) How does the equation $10^1 \cdot 10^{.1} = 10^{1.1}$ relate to exercise 11.

(15) Write an equation like that of exercise 14 to explain the relationships of exercise 13.

(16) Extend the table of exercises 1 to -1 by tenths ($n = -.1, -2, \ldots, -1.0$).

(17) How do the new entries relate to your other tables?

(18 - 25) Use your table of exercise 16 to calculate

(18) $10 \times .7943$ (19) $\sqrt[4]{.1585}$

(20) $25.1189 \div .1259$ (21) $(.5012)^3$

(22) $\sqrt{\dfrac{19.9526 \times 1.9953}{.1585}}$ (23) $(.2512)^{4/3}$

(24) $\dfrac{.1259 \times .5012}{.3981}$ (25) $\dfrac{31.6228 \times 19.9526}{\sqrt[3]{.1259}}$

2.9 Logarithms

The exponents you worked with in Section 2.8 are usually called logarithms or logs for short. In fact logarithms with base 10 are defined by the equation

$$\boxed{10^{\log N} = N}$$

Thus log N is the power to which ten must be raised to give N. More generally

$$\boxed{b^{\log_b N} = N}$$

In this case $\log_b N$ (read "log to the base b of N") is the exponent to which b is raised to give N.

EXAMPLES:

$$2^{\log_2 8} = 8 \quad \text{and} \quad \log_2 8 = 3 \quad \text{(Be sure to see why!)}$$

$$e^{\log_e 1} = 1 \quad \text{and} \quad \log_e 1 = 0$$

When the base is not noted it is understood to be 10.*

Examples: $\log 100 = 2$ (since $10^2 = 100$)

$\log .1 = -1$ (since $10^{-1} = .1$)

Another interpretation of logs is often useful. The following two equations are equivalent:

---

*The notation ln N is used for the natural log of N: $\ln N = \log_e N$ where $e = 2.7^+$: e is an important constant (like π) in advanced mathematics.

EXPONENTIAL FORM		LOGARITHMIC FORM
$b^{\ell} = p$	$\Leftrightarrow$	$\log_b p = \ell$

This translation between exponential form and logarithmic form is best remembered by a few key examples

Examples:

$$10^2 = 100 \Longleftrightarrow \log_{10} 100 = 2 \quad \text{or} \quad \log 100 = 2$$

$$2^3 = 8 \Longleftrightarrow \log_2 8 = 3$$

Two hints for remembering this relationship

1. Note how the base is the same

$$2^3 = 8 \qquad \log_2 8 = 3 \quad \text{(Recall log to the base 2 of 8 is 3.)}$$

base

2. The log equation says, "the log... is ...." Remember that a log is an exponent so the number following "is" will be the exponent

exponent or log

$$2^3 = 8 \qquad \log_2 8 = 3$$

Exercises:

(1 - 6) Using the defining equation $N = 10^{\log N}$, express each of the following as a power of 10.

Example: $5 = 10^{\log 5}$

(1) 12

(2) .07

(3) 13

(4) 2846

(5) $\frac{1}{2}$

(6) π

(7 - 15) Give x as a logarithm in each exercise

Example $5^x = 37$ $x = \log_5 37$

(7) $3^x = 27$

(8) $2^x = 16$

(9) $487^x = 1$

(10) $32^x = 2$

(11) $10^x = 1000$

(12) $10^x = .1$

(13) $10^x = 2$

(14) $10^x = 387$

(15) $10^x = 10$

(16) In exercises 7 - 15 give the value of x when you know it.
Example: $5^x = 25$ $x = \log_5 25 = 2$

(17 - 28) Write equivalent log equations for the following exponential equations

Example: $5^2 = 25$ $\log_5 25 = 2$

(17) $10^3 = 1000$

(18) $10^{-1} = .1$

(19) $2^5 = 32$

(20) $3^2 = 9$

(21) $3.7^{2.5} = 26.33$

(22) $10^{.3010} = 2$

(23) $10^{.4771} = 3$

(24) $10^{.7781} = 6$

(25) $10^{-.4771} = \frac{1}{3}$

(26) $10^{-2} = .01$

(27) $10^5 = 100{,}000$

(28) $e^2 = 7.39$ ($\log_e = \ln$)

2.10 Calculating with Logarithms

Three basic theorems are basic to calculation with logarithms.

I. $\log (xy) = \log x + \log y$

II. $\log \left(\frac{x}{y}\right) = \log x - \log y$

III. $\log x^p = p \log x$

Proofs of these important theorems are all based on the definition of logs

$$10^{\log N} = N.$$

Proof of I: $xy = \underline{10^{\log xy}}$, $x = 10^{\log x}$ and $y = 10^{\log y}$

$xy = (10^{\log x})(10^{\log y}) = \underline{10^{\log x + \log y}}$ by Exp Law (1) p. 2.1 - 2.

Since the underscored terms are each equal to xy:

$$10^{\log xy} = 10^{\log x + \log y}$$

Since the bases are the same the exponents are also equal:

$$\log xy = \log x + \log y$$

Proof of III: $x^p = \underline{10^{\log x^p}}$

$x^p = (10^{\log x})^p = \underline{10^{p \log x}}$ by Exp Law (3), p. 2.1 - 2.

Since the underscored terms are each equal to x^p:

$$10^{\log x^p} = 10^{p \log x}$$

Since the bases are equal the exponents are also equal:

$$\log x^p = p \log x.$$

Example:

$\log(3 \cdot 5) = \log 3 + \log 5$

$\log \frac{34.7}{23.4} = \log 34.7 - \log 23.4$

$\log (2.7)^7 = 7 \log 2.7$

To calculate with logs the following steps are followed:

(1) Find the logs of the numbers

(2) Calculate by the appropriate log techniques (using Theorems I - III)

(3) Restore the numerical answer by 10^x

Example: Calculate 38.47×56.14

$\log (38.47 \times 56.14) = \log 38.47 + \log 56.14$ (by I)

$= 1.5851 + 1.7493$ (by calculator)

$= 3.3344$ (by calculator)

$10^{3.3344} = 2159.7$ (by calculator)

Example: Calculate $(34.3)^{.347}$

$\log (34.3)^{.347} = .347 \log 34.3$ (by III)

$= .347 (1.5353) = .5327$ (by calculator)

$10^{.5327} = 3.4099$ (by calculator)

Exercises:

1. Use the method of proof of Theorem I to prove Theorem II.

2 -12. Calculate by using logs. Use your calculator only to process the logarithms and to calculate 10^x.

(2) 74.1×1.64

(3) $.163 \div 2.18$

(4) $(82.7)^{1.4}$

(5) $\sqrt[3]{34}$ (note: Use $34^{1/3}$)

(6) $\dfrac{38.5 \times 62.4}{71.8}$

(7) $\dfrac{143.6}{71.2 \times 84.7}$

(8) 23.7×41.3^2

(9) $\sqrt[5]{64.5} \times 81.2$

(10) $61.2 \div (43.6)^{1.3}$

(11) $45^{.6} \times 34^{.02}$

(12) π^{π}

2.11 Logarithmic Equations

In Section 2.10 we developed the basic properties of logarithms which may be summarized as follows:

		Logarithms
Multiplication	$\longrightarrow$	Addition
Division	$\longrightarrow$	Subtraction
Powers	$\longrightarrow$	Multiplication
Roots	$\longrightarrow$	Division

These properties may be used in translating algebraic equations into logarithmic equations.

Example. Express $a = \frac{b^2}{c}$ as a log equation.

$$\log a = 2 \log b - \log c$$

Example: Calculate by logs: $\frac{42.5 \times 37^3}{23^4}$

Solution steps:

1. Form an equation $x = \frac{42.5 \times 37^3}{23^4}$
2. Use log properties to write a logarithmic equation
   $\log x = \log 42.5 + 3 \log 37 - 4 \log 23$
3. Determine the logs and simplify
   $\log x = 1.6284 + 3(1.5682) - 4(1.3617) = .8861$
4. Return from logs to algebra by using 10^x (Recall $10^{\log x} = x$)* $x \doteq 7.69$ $(10^{.8861} = 7.69)$

Example: If $\log x = a$, express $\log\left(\frac{x}{100}\right)$ in terms of a

$$\log\left(\frac{x}{100}\right) = \log x - \log 100 = a - 2.$$

---

*The process of returning from logs is often called antilog. Thus antilog N is the same as 10^N and in this case antilog .8861 $\doteq$ 7.69.

Exercises:

Write logarithmic equations for each of the following:

(1) $x = \frac{35 \times 23}{267}$

(2) $x = 23^2 \sqrt{35}$

(3) $x = \frac{6720}{7.6 \times 14}$

(4) $\frac{41^3 \times 23^{2/3}}{\sqrt[4]{17}}$

(5) $a = b^2 c$

(6) $a = \frac{b}{c\sqrt{d}}$

7 - 14 For the following exercises let $\log x = a$, $\log y = b$, $\log z = c$. Express answers in terms of a, b and c.

(7) $\log xy$

(8) $\log \frac{x}{y}$

(9) $\log x^2$

(10) $\log \frac{xy}{z^2}$

(11) $\log (1000z)$

(12) $\log (.01y)$

(13) $\log x\sqrt{y}$

(14) $\log \sqrt{xy}$

15 - 22 Translate the given log equation into an algebraic equation. Simplify when possible.

(15) $\log x = \log 3 - \log 5$

(16) $\log x = \log 3 + \log 5$

(17) $\log x = 2 \log 5$

(18) $\log x = 2 + \log 5$ (See Ex. 11)

(19) $\log x = \frac{1}{2} \log 36$

(20) $\log x = 2 \log y + 3 \log z$

(21) $\log x - \log 100 = 3$

(22) $\log 100 - \log x = \log 5$

(23) If $\log x = a$, find antilog $2a$. (See footnote on page 2.11 - 1)

(24) If $\log x = a$, find antilog $(a + 2)$.

(25) If $\log x = a$ and $\log y = b$, find antilog $(2a - 3b)$.

26 - 30 Solve for x:

(26) $3^x = 30$

Solution: Write the log equation $x \log 3 = \log 30$

Solve for x $x = \frac{\log 30}{\log 3}$

(Note that this is division, not subtraction)

Use your calculator to find x $x \doteq \frac{1.4771}{.4771} \doteq 3.0960$

(27) $2^x = 10$

(28) $5^x = .5$

(29) $4^x = 21$

(30) $3.14^x = 5.12$

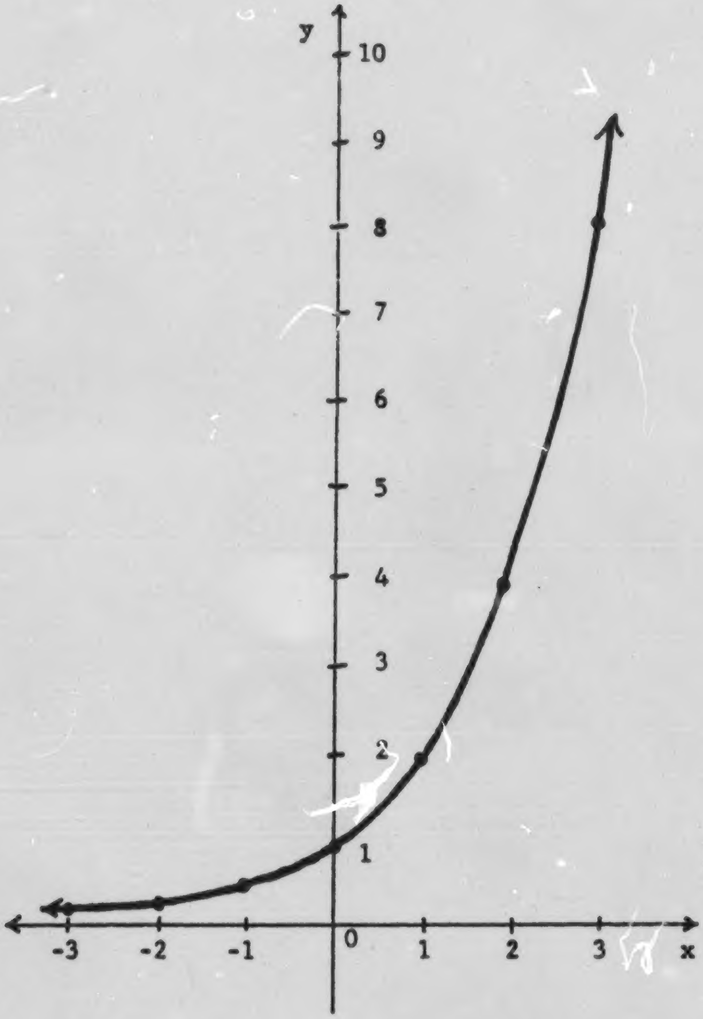
2.12 Graphs of Exponential and Logarithmic Functions

In the exercises for this section you will be asked to construct graphs for various exponential and logarithmic functions. This is easily accomplished by making a table of values (by using your calculator) and plotting the resulting points.

Here we suggest how you could sketch the graph of two related functions without use of the calculator:

Example: Sketch the graph of $y = 2^x$, for values of x in the range $-3 \leq x \leq 3$.

x	2^x
3	8
2	4
1	2
0	1
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$

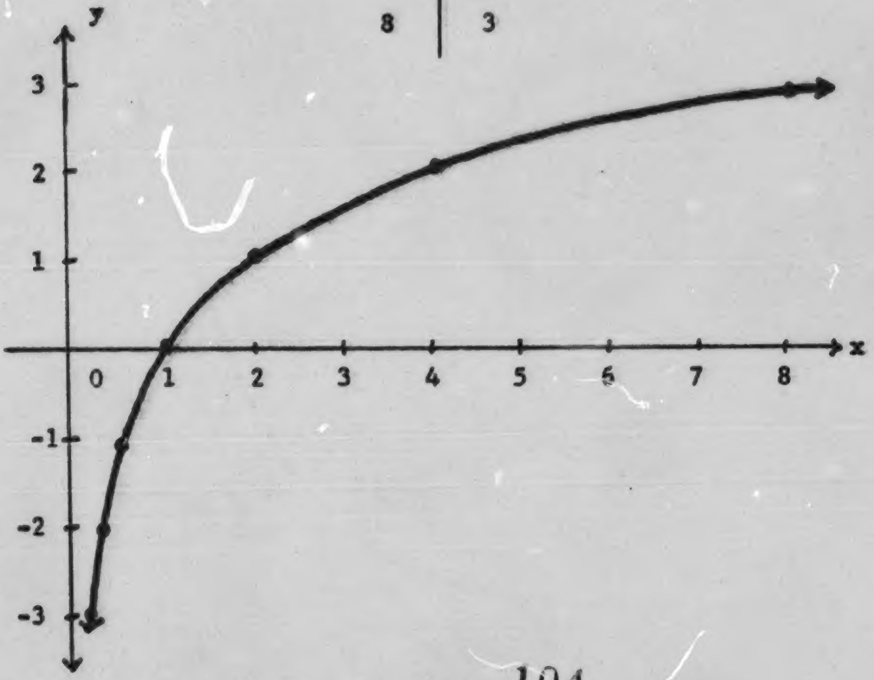


Example: Sketch the graph of $y = \log_2 x$ for values of x in the range $\frac{1}{8} \leq x \leq 8$

Solution: Recall that the equation $y = \log_2 x$ is equivalent to $2^y = x$.

Thus we can construct our table of values

x 2^y	$\log_2 x$ y
$\frac{1}{8}$	-3
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



Exercises:

1 - 5 Use your calculator to locate additional points for the graph $y = 2^x$. Check to see that they would lie on the graph on page 2.12 - 1.

(1) $(.5, 2^{.5})$ (Solution $2^{.5} = 1.4^+$. Is the point $(.5, 1.4^+)$ on the graph?)

(2) $(1.5, 2^{1.5})$ (3) $x = 2.5$

(4) $x = -.5$ (5) $x = -1.5$

6 - 10 Logate additional points for the graph $y = \log_2 x$. Check to see that they would lie on the graph on page 2.12 - 2.

(6) $(3, \log_2 3)$ (Solution $\log_2 3 = y$ translates to $2^y = 3$. You may find that by direct trial with your calculator or by the method of exercises 26 - 30 on p. 2.11 - 3.)

(7) $(5, \log_2 5)$ (8) $(6, \log_2 6)$

(9) $x = 7$ (10) $x = .7$

11 - 15 Set up axes on a sheet of graph paper with range $-2 \leq y \leq 10$ and domain $-2 \leq x \leq 10$

(11) Draw the graph of $y = 10^x$ for x-values in the domain $-2 \leq x \leq 1$. Plot at least ten values (determined by calculator) on your graph.

(12) Draw the graph of $y = \log x$ on the same graph sheet. Plot at least ten values (determined by calculator) on your graph.

(13) Draw the graph of $y = x$ on the same graph sheet.

(14) Crease your graph along the line $y = x$. What do you notice about the other two graphs?

(15) Explain why this relation holds.

(16) Sketch the graph $y = 3^x$ for x-values in the domain $-2 \leq x \leq 2$.

(17) Sketch the graph $y = \log_3 x$ for x values in the domain $\frac{1}{9} \leq x \leq 9$.

2.13 Chapter 2 - TEST

(1 - 10) Answer each of the following questions.

1) Find n when $\log n = 1 + \log 2$.

2) If $3^x = 9^y$, express x in terms of y.

3) If $\log_b 81 = 4/3$ find b.

4) Find the smallest integral value of x such that $(\frac{16}{15})^x > 10$.

5) If $\log_a 2 = b$ and $\log_a 3 = c$, express $\log_a 54$ in terms of b and c.

6) If $y = \log 5$, find the value of 10^{2y}.

7) Solve for y: $3^{2y+3} = \frac{1}{3}$.

8) Solve for x: $x^{-5/2} = 32$.

9) What is the domain of $y = \log x$?

10) Rewrite .0003472 in scientific notation.

(11 - 20) Match each question with the letter that best answers the question.

11) $3^0(3^{-1} \div 3^{-4})$

12) $(2.7)(10^{-1})$

13) $x^{x+.3} = 8;\ x =$

14) $[(.3)^{-6}]^{-1/2}$

15) $x^{-2/3} = 9;\ x =$

16) $27\sqrt{3} \div 3^{-1} \cdot \sqrt{3^3}$

17) $3^{3/2}$

18) $(-27)^{-2/3}$

19) $(0.000027)(10^6)$

20) $\frac{27}{27^{1/2}} \cdot 27^0 \cdot 3^{-3}$

A) $\pm\ \frac{1}{27}$

B) 27

C) 2.7

D) .27

E) .027

F) $\sqrt{27}$

G) None of the above

21) Find the exact value of $(345621)^2$.

(22 - 23) Choose one of the following two questions.

22 a) Graph the function $f(x) = 2.3^x$ for values of x between -3 and 3 inclusive. Label the graph with its equation.

b) In the same set of axis used in part (a) graph the function $g(x) = \log_{2.3} x$ for values of x between 0 and 3. Label the graph with its equation.

c) Using your graph approximate the value of $\log_{2.3} e$.

23 a) Graph the function $f(x) = \frac{1}{2}^{x}$ for values of x between -3 and 3 inclusive. Label the graph with its equation.

b) On the same set of axis used in part (a) graph the function $y(x) = \frac{1}{2}^{-x}$ for values of x between -3 and 3 inclusive. Label the graph with its equation.

c) Write a function that is the inverse of the function $f(x) = \frac{1}{2}^{x}$.

Chapter 3 TRIGONOMETRY OF THE RIGHT TRIANGLE

In this section you will review your understanding of the trigonometry of the right triangle and extend your knowledge to six trignometric functions.

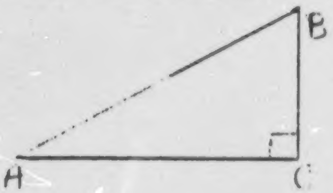
3.1 The Six Trignometric Functions For Acute Angles.

When you studied similar triangles you learned that there are special relationships between the measures of the acute angles of a right triangle and the lengths of the sides of the triangle. These ratios of the sides of a right triangle can be summarized as follows:

$$\text{sin (sin)} \angle A = \frac{\text{length of opposite leg } \overline{BC}}{\text{length of hypotenuse } \overline{AB}}$$

$$\text{cosine (cos)} \angle A = \frac{\text{length of adjacent leg } \overline{AC}}{\text{length of hypotenuse } \overline{AB}}$$

$$\text{Tangent (tan)} \angle A = \frac{\text{length of opposite leg } \overline{BC}}{\text{length of adjacent leg } \overline{AC}}$$



where $\triangle$ ABC has a right angle at vertex C as illustrated in the diagram.

Thus $\sin \angle A = \frac{BC}{AB}$[*] $\qquad \sin \angle B = \frac{AC}{AB}$

$\cos \angle A = \frac{AC}{AB} \qquad \cos \angle B = \frac{BC}{AB}$

$\tan \angle A = \frac{BC}{AC} \qquad \tan \angle B = \frac{AC}{BC}$.

Notice that the values of each function are different for each acute angle of the right triangle. This is because the functions are defined by the relative phrases opposite leg and adjacent leg. Many students remember the definitions of these functions by the mnemonic:

---

[*] We will use the notation BC to represent the length of the line segment from B to C, ($\overline{BC}$). Similarly $m \angle A$ represents the measure of the angle whose vertex is at point A.

ALGEBRA
OSCAR

Oscar
Had
A
Headache
Over
Algebra

$$\sin \angle = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$\cos \angle = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$

$$\tan \angle = \frac{\text{opposite leg}}{\text{adjacent leg}}$$

If these ratios sin, cos and tan are inverted then we can define three new ratios that are called cosecant (csc), secant (sec) and cotangent (cot) respectively.

thus

$$\csc \angle A = \frac{AB}{BC} = \frac{1}{\sin \angle A} = \frac{\text{hypotenuse}}{\text{opposite leg}}$$

$$\sec \angle A = \frac{AB}{AC} = \frac{1}{\cos \angle A} = \frac{\text{hypotenuse}}{\text{adjacent leg}}$$

$$\cot \angle A = \frac{AC}{AB} = \frac{1}{\tan \angle A} = \frac{\text{adjacent leg}}{\text{opposite leg}}$$

In this section we are dealing only with acute and right angles. We will study other kinds of angles in future sections. Fill in the following charts by using the $\boxed{\sin}$, $\boxed{\cos}$ and $\boxed{\tan}$ keys on your calculator. Fix your calculator to 4 decimal places. You will be asked to recognize special properties of these functions in the exercises at the end of this section.

$\angle$	sin
0°	
1°	
15°	
30°	
45°	.7071
60°	
75°	
89°	
90°	

$\angle$	cos
0°	
1°	
15°	.9659
30°	
45°	
60°	
75°	
89°	
90°	

$\angle$	tan
0°	
1°	
15°	
30°	
45°	
60°	
75°	
89°	57.2900
90°	

Your calculator probably does not have special keys to determine cosecant (csc), secant (sec) and cotangent (cot).* These values can be determined by using the fact that

$$\left.\begin{array}{l}\text{csc and sin}\\ \text{sec and cos}\\ \text{cot and tan}\end{array}\right\}\quad\text{are reciprocals.}$$

To find csc 1° you should find sin 1° and then key $\boxed{\frac{1}{x}}$, thus 57.2987 = csc 1°. Complete the following tables:

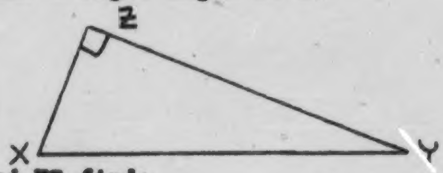
∡	csc ∡
0°	
1°	57.2987
15°	
30°	
45°	
60°	
75°	
89°	1.0002
90°	

∡	sec ∡
0°	
1°	
15°	
30°	1.1547
45°	
60°	
75°	
89°	
90°	error

∡	cot∡
0°	
1°	
15°	
30°	
45°	
60°	
75°	0.2679
89°	
90°	

Exercise set 3.1

1 - 8 Triangle XYZ has a right angle at Z.



In terms of XY, YZ and XZ find:

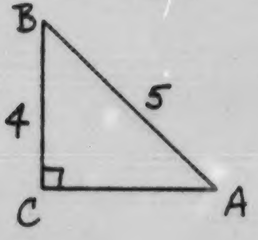
1) sin ∡ X =
2) cos ∡ Y =
3) cot ∡ Y =
4) tan ∡ X =
5) sec ∡ X =
6) csc ∡ X =
7) sin ∡ Y =
8) sec ∡ Y =

---

*Note ctn = cot alt abbreviation

9 - 12 In each of the following $\measuredangle A$ is an acute angle of right triangle ABC having right angle at C. Find the values of each of the other five functions and sketch a diagram of each triangle indicating the length of each side.

Example: $\sin \measuredangle A = \frac{4}{5}$



by the Pythagorean Relation

$$(AC)^2 + (BC)^2 = (AB)^2$$
$$(AC)^2 + 4^2 = 5^2$$
$$(AC)^2 + 16 = 25$$
$$(AC)^2 = 9$$
$$AC = 3$$

So

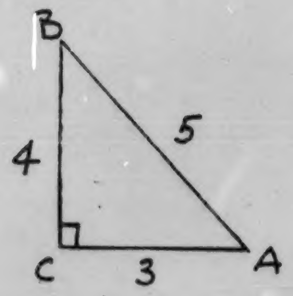
$\cos \measuredangle A = \frac{3}{5}$

$\tan \measuredangle A = \frac{4}{3}$

$\cot \measuredangle A = \frac{3}{4}$

$\sec \measuredangle A = \frac{5}{3}$

$\csc \measuredangle A = \frac{5}{4}$



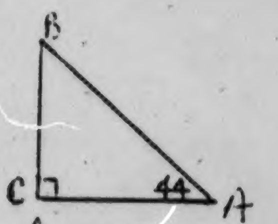
9) $\cos \measuredangle A = \frac{8}{17}$

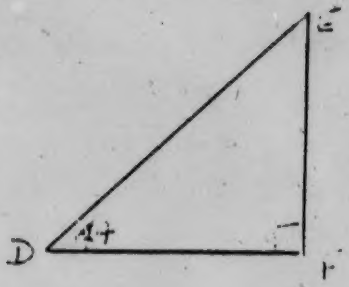
10) $\tan \measuredangle A = 2.4$

11) $\sin \measuredangle A = \frac{1}{2}$

12) $\cot \measuredangle A = t$

13 - 17 $\triangle ABC$ and $\triangle DEF$ are both right triangles. $m\measuredangle A = 44$ and $m\measuredangle D = 44$.





13) Why is $\triangle ABC$ similar to $\triangle DEF$?

14) Find $\sin \angle B$ and $\sin \angle E$. Why are these the same or different?

15) If $AC = 10.00$, $BC = 9.66$ and $DF = 3.00$, find EF.

16) Using your results in (15) find AB and DE.

17) Find AB and DE by using a method different from the method you used in (16).

Look back at the tables that you filled in earlier in this section. We would like to make some observations.

Example: From your tables you should notice that the same values appear for the sin and cos.

$$\sin \angle A = \frac{BC}{AB} = \cos \angle B$$

$$\sin \angle B = \frac{AC}{AB} = \cos \angle A$$

Thus $\sin x^\circ = \cos (90 - x)^\circ$. For example, $\sin 1^\circ = \cos 89^\circ$ and $\sin 45^\circ = \cos 45^\circ$.

18 - 19 Answer each of the following questions and verify your responses by a method similar to the one used in the example above.

18) Why is $\csc 30^\circ = \sec 60^\circ$?

19) Why is $\tan 15^\circ = \cot 75^\circ$?

20) Find each of the following:

$$\frac{\sin 0^\circ}{\cos 0^\circ} =$$

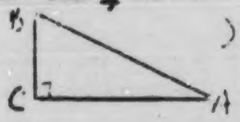
$$\frac{\sin 1^\circ}{\cos 1^\circ} =$$

$$\frac{\sin 15^\circ}{\cos 15^\circ} =$$

$$\frac{\sin 30^\circ}{\cos 30^\circ} =$$

Now compare your answers with the entries in your tables. Make a guess about $\frac{\sin 75^\circ}{\cos 75^\circ}$.

What is the relation between $\sin \angle A$, $\cos \angle A$ and $\tan \angle A$? Why is this true? (Hint: use a triangle such as:



)

21) What do you guess is true about csc, sec and cot? Support your conjecture by using a method similar to that in exercise 20.

You know that the hypotenuse is always the longest side of a right triangle. Therefore the sin and cos of any acute angle must be less than one because in each of these ratios the numerator is smaller than the denominator.

Example:

$$\sin \angle A = \frac{BC}{AB}$$

$$\frac{BC}{AB} < 1 \text{ because } BC < AB.$$

22 - 27 Using arguments similar to the preceding example, explain each of the following:

22) The cosecant of any acute angle is always greater than 1.

23) The secant of any acute angle is always greater than 1.

24) The tangent of an acute angle is greater than 1.

25) The value of the sine of an acute angle increases as the measure of the angle increases.

26) The value of the cosine of an acute angle decreases as the measure of the angle increases.

27) The value of the tangent of an acute angle increases faster than the sine of an acute angle.

28 - 29 You have probably noticed that several entries in your table are rather unusual.

$\csc 0^\circ$ = error message
$\cot 0^\circ$ = error message
$\tan 90^\circ$ = error message
$\sec 90^\circ$ = error message

28) Make a conjecture about why you get these error messages on your calculator.

29) Make a conjecture about why $\sin 0^\circ = 0$ and $\cos 0^\circ = 1$.

30 - 35 Decide whether each of the following is true or false.*

30) If $\measuredangle A$ is an acute angle then $\sin \measuredangle A + \cos \measuredangle A > 1$.

31) The cosine of an acute angle is always greater than the sine of the same angle.

32) If $\measuredangle A$ and $\measuredangle B$ are acute angles in a right triangle then $\sin \measuredangle A - \cos \measuredangle B = 0$.

33) If $\measuredangle A$ and $\measuredangle B$ are acute angles and $m\measuredangle A < m\measuredangle B$ then $\tan \measuredangle A < \tan \measuredangle B$.

34) If $\sin \measuredangle A = \cos \measuredangle A$ and $\measuredangle A$ is acute then $m\measuredangle A = 45^\circ$.

35) If $\measuredangle A$ and $\measuredangle B$ are acute angles in a right triangle then $\sin \measuredangle A \cdot \cos \measuredangle B = 1$.

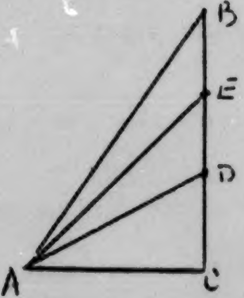
---

*You may wish to test specific values with your calculator but if you answer "true" you must assure yourself that the statement is true over the entire range of values.

3.2 Variations of the Trigonometric Functions:

As an angle changes size, the values of any of the six trigonometric functions also change.

The figure below contains three right triangles. AC is a leg of each of these triangles.



$m\angle DAC < m\angle EAC < m\angle BAC.$

$\tan\angle DAC = \frac{DC}{AC}$

$\tan\angle EAC = \frac{EC}{AC}$

$\tan\angle BAC = \frac{BC}{AC}$

and $DC < EC < BC$ so $\tan\angle DAC < \tan\angle EAC < \tan\angle BAC$. So you probably recognized from the tables that you completed in section 3.1 that as an acute angle increases its tangent increases also. When the measure of an angle is very close to zero the length of the side opposite that angle is also very small so its tangent is very close to zero. If a triangle could have an angle whose measure was zero the side opposite would be zero so its tangent would be zero. When the angle has a measure of 45° as $\angle EAC$ then the triangle is isosceles and $EC = AC$ so $\tan 45^\circ = \frac{EC}{AC} = 1$.

If the angle at A had a measure close to 90° then the side opposite it would be very long while $\overline{AC}$ would remain constant. Hence the tangent of such an angle would be very large. Complete the following equations by using your calculator (set to 2 decimal places.)

$\tan 80^\circ =$
$\tan 85^\circ =$
$\tan 89^\circ =$
$\tan 89.5^\circ =$
$\tan 89.75^\circ =$
$\tan 89.95^\circ =$
$\tan 89.99^\circ =$
$\tan 89.999^\circ =$
$\tan 90^\circ =$

Your calculator probably gives you an error message for tan 90°. We will discuss why this is true in future sections that deal with angles that are not acute.

In the figure below $\overline{AB}$, $\overline{AD}$, $\overline{AF}$ and $\overline{AH}$ are successive positions of the hypotenuse as an angle changes in size. Each of these has the same length as they are radii of the same circle whose center is at A.

$\sin \measuredangle BAC = \frac{BC}{AB}$

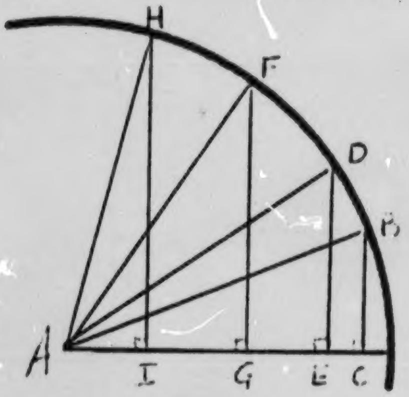
$\sin \measuredangle DAE = \frac{DE}{AD}$

$\sin \measuredangle FAG = \frac{FG}{AF}$

$\sin \measuredangle HAI = \frac{HI}{AH}$

and $AB = AD = AF = AH$ and

$BC < DE < FG < HI$



So $\sin \measuredangle BAC < \sin \measuredangle DAE < \sin \measuredangle FAG < \sin \measuredangle HAI$.

When the measure of the angle is very near zero the length of the opposite side is very near zero. When the measure of the angle is very near 90° the length of the opposite side is very near the length of the hypotenuse so the value of the sine ratio is very nearly one.

Exercise set 3.2

Use the diagram above to answer each of the following questions.

1) $\cos \measuredangle BAC =$ 2) $\cos \measuredangle DAE =$

3) $\cos \measuredangle FAG =$ 4) $\cos \measuredangle HAI =$

5) What inequality can you write among AI, AG, AE and AC? AI__ AG___AE ___ AC.

6) If the measure of an angle is near zero its cosine must be near _____. Why?

7) If the measure of an angle is near 90° its cosine must be near _____. Why?

8) Why does the cosine of an acute angle decrease as the measure of the angle increases?

Complete each of the following inequalities using < or > . Use your calculators, if you wish.

9) $\frac{1}{2}$ ___ $\frac{3}{4}$ and $\frac{2}{1}$ ___ $\frac{4}{3}$

10) 2 ___ 5 and $\frac{1}{2}$ ___ $\frac{1}{5}$

11) $\frac{5}{7}$ ___ $\frac{2}{3}$ and $\frac{7}{5}$ ___ $\frac{3}{2}$

12) .256 ___ .583 and $\frac{1}{.256}$ ___ $\frac{1}{.583}$

13) If $\frac{x}{y} < \frac{a}{b}$ then $\frac{y}{x}$ ___ $\frac{b}{a}$, when x, y, a and b are positive. Why?

You know about the reciprocal relationships between pairs of trigonometric functions. Complete the following:

14) ______ = $\frac{1}{\csc x}$

15) ______ = $\frac{1}{\sec x}$

16) ______ = $\frac{1}{\cot x}$.

Complete each of the following inequalities using < or > .

17) $\sin 10 < \sin 70$ so $\csc 10$ ______ $\csc 70$.

18) $\sec 50 < \sec 85$ so $\cos 50$ ______ $\cos 85$.

19) $\tan 10 < \tan 30$ so $\cot 10$ ______ $\cot 30$.

Complete each of the following using the words increases or decreases.

20) As the measure of an acute angle increases, the value of its cotangent ______________.

21) As the measure of an acute angle decreases, the value of its secant ______________.

22) As the measure of an acute angle increases the value of its cosine ______________.

23) As the measure of an acute angle decreases, the value of its sine ______________.

24) As the measure of an acute angle decreases, the value of its tangent ______________.

25) As the measure of an acute angle increases, the value of its cosecant ______________.

26) Complete the following table:

function	value near 0°	behavior 0° to 90°	value near 90°
sine	near 0		
cosine			near 0
tangent		increases	
cosecant		decreases	near 1
secant			very large
cotangent	very large		

(27 - 30) Fill each of the following blanks with either 0 or 1.

27) The sine of an acute angle is always greater than _______ and less than __________.

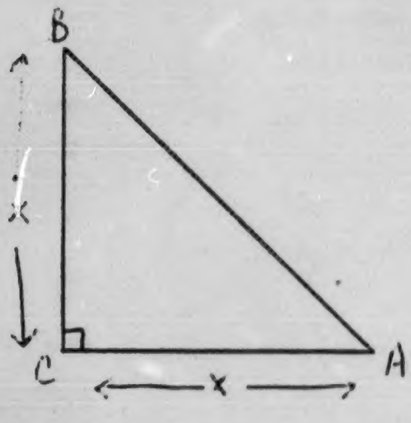
28) The cosine of an acute angle is always greater than __________ and less than __________.

29) The tangent and cotangent of an acute angle is always greater than ______________.

30) The secant and cosecant of an acute angle is always greater than ______________.

3.3 Special Right Triangles

In an isosceles right triangle the legs are congruent. In $\triangle ABC$, $\overline{AC} \cong \overline{BC}$.

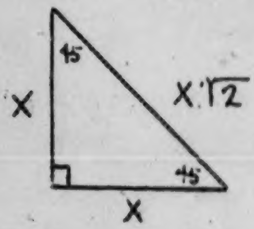


So $m\angle A = 45$ and $m\angle B = 45$.

let $AC = BC = X$. By the Pythagorean relation,

$$\begin{aligned}
(AC)^2 + (BC)^2 &= (AB)^2 \\
x^2 + x^2 &= (AB)^2 && \text{(substitution)} \\
2x^2 &= (AB)^2 && \text{(addition)} \\
\sqrt{2x^2} &= AB && \left(\begin{array}{l}\text{taking the square}\\ \text{root of each side}\end{array}\right) \\
x\sqrt{2} &= AB && \left(\begin{array}{l}\text{simplification}\\ \text{of radicals}\end{array}\right)
\end{aligned}$$

Hence the lengths of the sides of any isosceles triangle* can be represented as



and

$$\sin 45 = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45 = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45 = \frac{x}{x} = 1$$

$$\cot 45 = \frac{x}{x} = 1$$

$$\sec 45 = \frac{x\sqrt{2}}{x} = \sqrt{2}$$

$$\csc 45 = \frac{x\sqrt{2}}{x} = \sqrt{2}$$

---

*Notice that in any isosceles right triangle the legs are congruent and the length of the hypotenuse is the length of a leg times $\sqrt{2}$.

Find each of the following, using your calculator. Remember the reciprocal relationships.

sin 45 =

cos 45 =

tan 45 =

cot 45 =

sec 45 =

csc 45 =

Notice that your answers on your calculator are represented as decimals and not as radicals. The calculator values that you have found are rational approximations of irrational numbers. It is impossible to represent an irrational number such as a radical ($\sqrt{2}$ or $\frac{\sqrt{2}}{2}$) by a terminating or repeating decimal. Most of the trignometric values are irrational numbers so the best we can do is get an approximation. This is not a serious defect because generally our computations are accurate enough. Furthermore calculators do some nice rounding for us.*

Example:

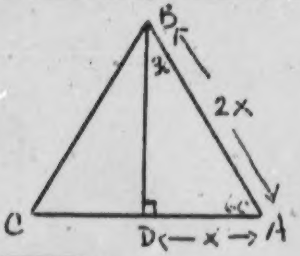
$$\frac{9 \sec 45}{\csc 45} = \frac{9(\sqrt{2})}{\sqrt{2}} = 9 \qquad \text{(algebraically)}$$

$$\frac{9 \sec 45}{\csc 45} = \frac{12.7279}{1.41421} = 9.0000 \text{(calculator)}$$

even though the intermediate calculator step,

$$\frac{12.7279}{1.4142} \neq 9. \qquad \text{(Why?)}$$

Remember that a 30-60-90 triangle is obtained when an altitude (angle bisector, median) is put in an equilateral triangle.



In equilateral triangle ABC
let AD = X, then AB = 2X and
again by the Pythagorean relation

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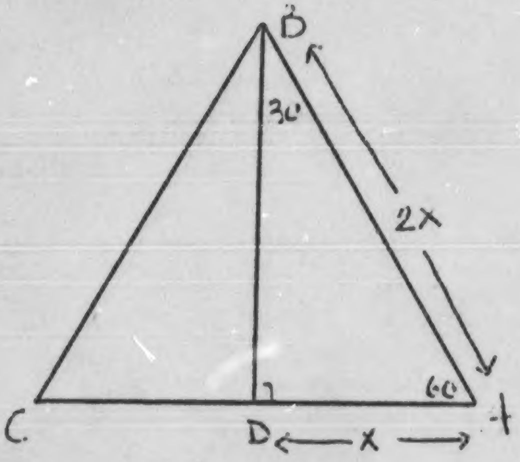
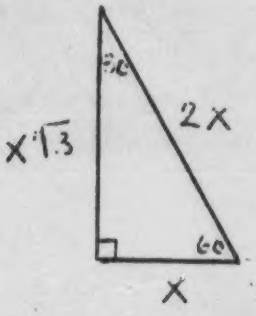
* Also, calculators generally work with more digits than they display.



$$\begin{aligned}
(AD)^2 + (BD)^2 &= (AB)^2 \\
x^2 + (BD)^2 &= (2x)^2 && \text{(substitution)} \\
x^2 + (BD)^2 &= 4x^2 && \text{(removing parenthesis)} \\
(BD)^2 &= 3x^2 && \text{(subtraction)} \\
BD &= \sqrt{3x^2} && \text{taking the square root of both sides} \\
BD &= x\sqrt{3} && \text{(simplifying radicals)}
\end{aligned}$$

Hence the lengths of the sides of any 30-60-90 triangle* can be represented as



and

$$\sin 30^\circ = \frac{x}{2x} = \frac{1}{2} \qquad \sin 60^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{x}{2x} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \tan 60^\circ = \frac{x\sqrt{3}}{x} = \sqrt{3}$$

$$\cot 30^\circ = \frac{x\sqrt{3}}{x} = \sqrt{3} \qquad \cot 60^\circ = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec 30^\circ = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \qquad \sec 60^\circ = \frac{2x}{x} = 2$$

$$\csc 30^\circ = \frac{2x}{x} = 2 \qquad \csc 60^\circ = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

---

*Notice that in any 30-60-90 triangle the length of the hypotenuse is twice the length of the short leg (opposite the 30° angle) and the length of the long leg (opposite the 60° angle) is the length of the short leg times $\sqrt{3}$.

Complete the following table using your calculator:

	sin	cos	tan	cot	sec	csc
30						
60						

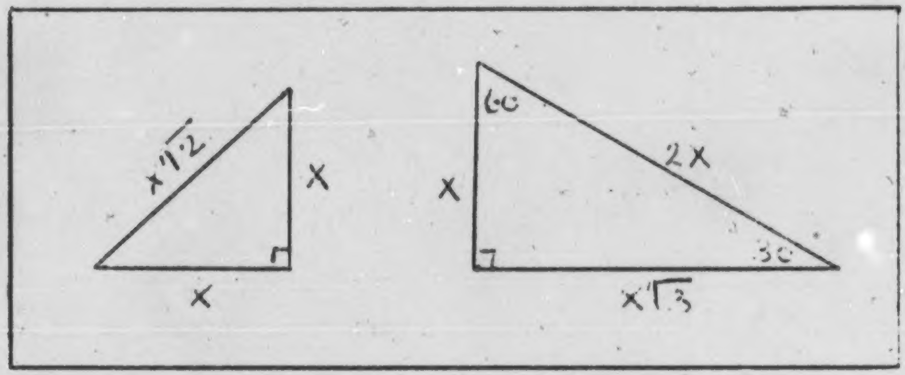
Compare these results to the values that we found above. Verify that the values are the good approximations.

Example:

$$\tan 30^\circ \doteq .5774 \qquad , \quad \frac{\sqrt{3}}{3} = \frac{1.7321}{3} \doteq .5774$$

Remember, we are again dealing with rational approximations of irrational numbers.

You should know the values of the functions for these particular angles because they occur frequently.* Do not try to memorize tables. If you do you will probably either get them mixed up or drive yourself crazy. An easier (and more sane) way to reproduce these values is to draw triangles.



Exercises 3.3

For each of the following:

(A) determine your answer algebraically (don't use the charts in this section)

(B) determine your answer on your calculator

(C) verify that (B) is a good approximation for (A)

---

*It is also important and convenient to know $\sqrt{3} \doteq 1.7321$, $\frac{\sqrt{3}}{2} \doteq .8660$
$\sqrt{2} \doteq 1.4142$ and $\frac{\sqrt{2}}{2} \doteq .7071$.

Example:

$$\cos 45^\circ + \tan 45^\circ$$

(A) $\frac{\sqrt{2}}{2} + 1 = \frac{\sqrt{2}}{2} + \frac{2}{2} = \frac{2 + \sqrt{2}}{2}$

(B) $.7071 + 1.000 = 1.7071$

(C) $\frac{2 + \sqrt{2}}{2} = \frac{3.4142}{2} = 1.7071$

(1) sin 30 + cos 45

(2) sin 30 cos 60 + cos 30 sin 60*

(3) 1 + tan 45

(4) 2 cos 30 + 3 csc 30

(5) $\cos^2 30 + \sin^2 30$**

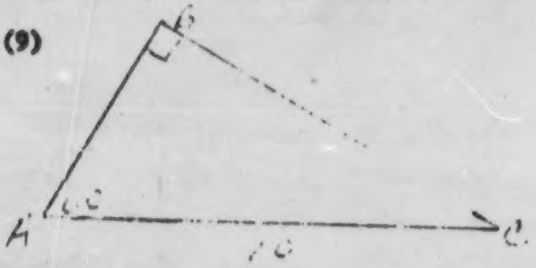
(6) 2 cos 45 - 3 cot 60

(7) sec 45 - 2 cos 60

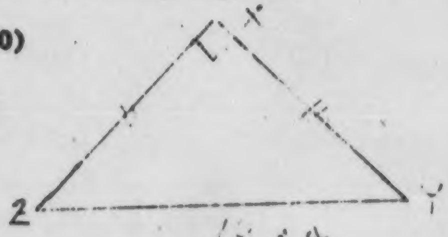
(8) sec 30 + csc 30

Find all the sides and angles of each of the following figures.

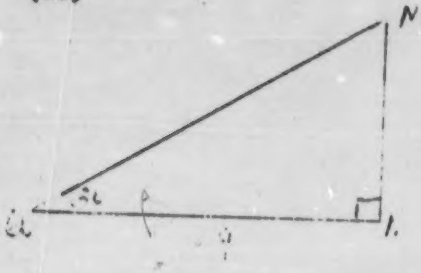
(9)



(10)



(11)



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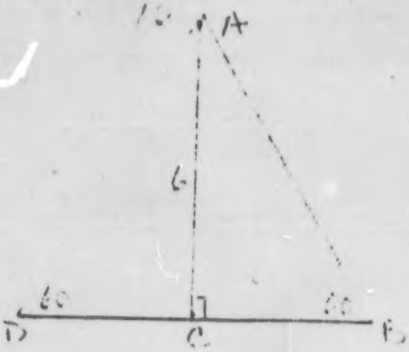
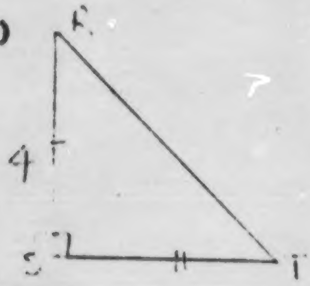
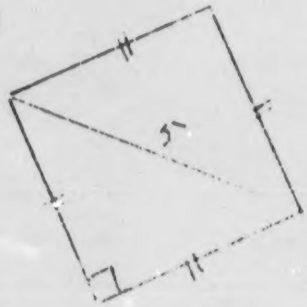


(13)



(14)



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*sin 30 cos 60 means (sin 30) times (cos 60)

**Recall that $\cos^2 30$ means $(\cos 30)^2$ or (cos 30) times (cos 30)

3.4 Measuring Angles

We have been expressing the measures of angles as degrees in decimals. Appropriately, these units of measure are called decimal degrees. This has been compatible with the way most calculators deal with the measure of angles. Very often the measure of an angle is written in other units that are expressed as degrees - minutes - seconds.

60 minutes = 1 degree $(60' = 1^\circ)$
60 seconds = 1 minute $(60'' = 1')$

Thus, $35.5^\circ = 35^\circ\ 30'$ and $47.26^\circ = 47^\circ\ 15'\ 36''$. You probably can easily verify the first equation above because .5 of a degree is clearly 30 minutes. The second equation requires more careful analysis.

Example 1: Convert 47.26° to degrees-minutes-seconds.

$$1^\circ = 60' = 3600''$$

$$.26^\circ = .26(3600) = 936''$$

$$936'' = 15'\ 36''$$

[936 divided by 60 has a quotient of 15 and a remainder of 36]

so $47.26^\circ = 45^\circ\ 15'\ 36''$.

Example 2: Convert $53^\circ\ 14'\ 28''$ to decimal degrees

$$14' = 14(60) = 840''$$

$$14'\ 28'' = 840 + 28 = 868''$$

$$\frac{3600''}{1^\circ} = \frac{868''}{x}$$

$$3600\,x = 868$$

$$x = .24\overline{11}$$

Thus $53^\circ\ 14'\ 28'' = 53.24\overline{11}\,^\circ$

If you find the preceding examples particularly tedious* then you will be especially happy to learn that most scientific calculators have special keys that make these conversions in two keystrokes. We will now consider these examples again by using specific calculators.

Example 3: Convert 47.26° to degrees, minutes and seconds

	HP 33	TI 57	
(display)	47.26	47.26	(display)
	$\boxed{\text{f}}$ $\boxed{\rightarrow\text{H.MS}}$	$\boxed{\text{INV}}$ $\boxed{\text{2nd}}$ $\boxed{\text{D. MS}}$	
(display)	47.1536	47.1536	(display)

which means 47° 15′ 36″

Example 4: Convert 53° 14′ 28″ to decimal degrees

	HP 33	TI 57	
(display)	53.1428	53.1428	(display)
	$\boxed{\text{g}}$ $\boxed{\rightarrow\text{H}}$	$\boxed{\text{2nd}}$ $\boxed{\text{D.MS}}$	
(display)	53.2411	53.24111111	

which means 53.2411°

A careful look at the similarities and differences of these examples can help us understand how these calculators work and also some of the mathematics involved. The HP-33 keys $\boxed{\rightarrow\text{H.MS}}$ and $\boxed{\rightarrow\text{H}}$ are both on the $\boxed{6}$ key. The TI-58 uses the additional keystroke $\boxed{\text{INV}}$ in example 1. Notice that:

- converting from decimal degrees to degrees, minutes and seconds

---

* Make sure that you understand the mathematics used in each of these examples. They are tedious but the ideas are not particularly difficult.

— converting from degrees, minutes and seconds to decimal degrees

are inverse operations, so the logic exhibited by your calculator is reasonable.

On both of these calculators (and indeed on most scientific calculators) the decimal degree format is DDD.dd where DDD represents the integer portion of the angle and .dd denotes the fractional portion written as a decimal. The degree-minute-second format uses DDD.MMSSsss where again DDD represents whole degrees, MM represents minutes, SS represents seconds and sss represents fractional seconds. Observe that in either case the decimal point separates the degrees from the minutes and seconds.

If you have not already guessed H.MS represents hours-minutes-seconds. This is the same kind of measure as degrees-minutes-seconds, so you can use these conversions to change ordinary time (in hours-minutes-seconds) to decimal hours.

Example 5: 3 hours 15 minutes = 3.25 hours
3.75 hours = 3 hours 45 minutes.

Angles can be measured in units other than degrees*. Recall that

$$\boxed{1^\circ = \frac{1}{360} \text{ of a revolution}}$$

Another unit for measuring angles often used by engineers and scientists is called a grad.

$$\boxed{1 \text{ grad}^{**} = \frac{1}{400} \text{ of a revolution}}$$

thus a right angle = 100 grads = 90°.

---

*Still another unit called a radian will be discussed later.

**There is no special symbol that represents grads.

Most scientific calculators do not convert degrees to grads or grads to degrees by using a special key. We can develop a conversion by using a proportion.

$$\frac{\text{degrees}}{\text{grads}} = \frac{360}{400} = \frac{9}{10} = \frac{.9}{1}$$

and

$$\frac{\text{grads}}{\text{degrees}} = \frac{400}{360} = \frac{10}{9} = \frac{1}{.9}$$

> To convert from degrees to grads divide by .9
>
> To convert from grads to degrees multiply by .9

Example 6:

$$90^\circ = 90\ (\div .9) = 100 \text{ grads}$$
$$150 \text{ grads} = 150\ (\times .9) = 135^\circ$$

Calculators "wake-up" to a decimal degree format. To change to a grad format press:

HP- 33	TI 57
$\boxed{\text{g}}$ $\boxed{\text{GRD}}$	$\boxed{\text{2nd}}$ $\boxed{\text{Grad}}$

To change back to a degree format press:

HP- 33	TI 57
$\boxed{\text{g}}$ $\boxed{\text{DEG}}$	$\boxed{\text{2nd}}$ $\boxed{\text{Deg}}$

Example 7:

Find (a) cos 75 grads then (b) find cos 67.5°

	HP- 33		TI 57	
	ON	calculators "wake-up"	ON	
(display)	75		75	(display)
	$\boxed{\text{g}}$ $\boxed{\text{GRD}}$		$\boxed{\text{2nd}}$ $\boxed{\text{Grad}}$	
(display)	75.0000		75	(display)
	$\boxed{\text{f}}$ $\boxed{\text{cos}}$		$\boxed{\text{2nd}}$ $\boxed{\text{cos}}$	
(display)	.3827		.3826834324	(display)

answers to (a)

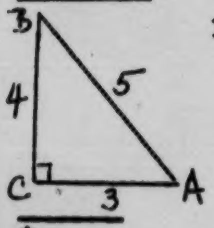
(display)	67.5	67.5	(display)
	$\boxed{\text{g}}$ $\boxed{\text{DEG}}$	$\boxed{\text{2nd}}$ $\boxed{\text{Deg}}$	
(display)	67.5000	67.5	(display)
	$\boxed{\text{f}}$ $\boxed{\text{cos}}$	$\boxed{\text{2nd}}$ $\boxed{\text{cos}}$	
(display)	.3827	.3826834324	(display)

answers to (b).

The angular mode (degrees or grads) has absolutely no effect on calculations of this type. Selecting the angular mode is easy to do and easy to forget so be careful and keep track of the mode being used.

Another way that angles can be measured is indirectly.

Example 8:



In right triangle ABC, AB = 5, BC = 4 and AC = 3.

Find m $\measuredangle$ A and m $\measuredangle$ B.

tan $\measuredangle$ A = $\frac{4}{3}$*

---

*The sine and cosine ratios can also be used. sin $\measuredangle$ A = $\frac{4}{5}$ = .2 and cos $\measuredangle$ A = $\frac{3}{5}$ = .6

$$\tan \measuredangle A = 1.\overline{33}$$
$$m \measuredangle A = 53.13^\circ$$

Since $\measuredangle A$ and $\measuredangle B$ are complementary $m \measuredangle B = (90 - 53.13^\circ) = 36.87^\circ$ *.

Notice that we are now using the trignometric ratios in a different way than we used them before. Previously we knew the measure of an angle and wanted to find the value of a particular function. Now we know the value of a particular function and are interested in knowing the measure of an acute angle. Again we are dealing with the mathematical concept of inverses. Several different kinds of notation represent this same idea. The inverses** of the trignometric functions are represented by the chart below.

function	inverse using arc	inverse using negative exponent
$y = \sin x$	$x = \text{arc} \sin y$	$x = \sin^{-1} y$
$y = \cos x$	$x = \text{arc} \cos y$	$x = \cos^{-1} y$
$y = \tan x$	$x = \text{arc} \tan y$	$x = \tan^{-1} y$
$y = \cot x$	$x = \text{arc} \cot y$	$x = \cot^{-1} y$
$y = \sec x$	$x = \text{arc} \sec y$	$x = \sec^{-1} y$
$y = \csc x$	$x = \text{arc} \csc y$	$x = \csc^{-1} y$

Both of these notations for inverses are used frequently so you need to be familiar with both of them.

If we look back at Example 8 we could have written

$$m \measuredangle A = 53.13^\circ$$

as $\text{arc} \tan A = 53.13^\circ$ or $\tan^{-1} A = 53.13^\circ$

---

*The trignometric ratios can be used here also.

**Technically, the trignometric functions do not have inverses. We will deal with this issue in future sections.

The following keystroke sequences show how the HP-33 and TI 57 can be used to do this example:

HP-33	TI 57
4 $\boxed{\text{ENTER}}$ 3 $\boxed{\div}$	4 $\boxed{\div}$ 3 $\boxed{=}$
(display) 1.3333*	1.333333333 (display)
$\boxed{\text{g}}$ $\boxed{\tan^{-1}}$	$\boxed{\text{INV}}$ $\boxed{\text{2nd}}$ $\boxed{\tan}$
(display) 53.1301	53.13010235 (display)
90 $\boxed{x \gtrless y}$	$\boxed{+/-}$
(display) 53.1301	-53.13010235 (display)
$\boxed{-}$	$\boxed{+}$ 90 $\boxed{=}$
36.8699	36.86989765

Exercises 3.4

(1 - 6) Convert each of the following angle measures to degrees-minutes-seconds. Do each (a) on your calculator (b) by pencil and paper.

(1) 45.12°
(2) 39.755°
(3) 87.215°
(4) 51.0375°
(5) 50.5 grads
(6) 13.5 grads

(7 - 12) Convert each of the following angle measures to decimal degrees. Do each (a) on your calculator and (b) by pencil and paper.

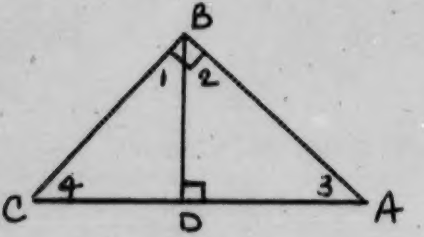
(7) $14^\circ\ 10'\ 30''$
(8) $68^\circ\ 23'\ 15''$
(9) $82^\circ\ 5'$
(10) $70^\circ\ 30'\ 18''$
(11) 90.25 grads
(12) $48.\overline{33}$ grads

---

*Most scientific calculators store more places than they display. In the case of the HP-33 more than four digits are carried throughout this entire computation.

(13) In Example 8 find $m\angle A$ and $m\angle B$ by using a method different from the method used in the example.

(14)



$\angle ABC$ and $\angle BDA$ are right angles

$BD = 5$ and DA 12

find $m\angle 1$, $m\angle 2$, $m\angle 3$, $m\angle 4$

(15 - 18) Find x

(15) $x = \sin 93$ grads

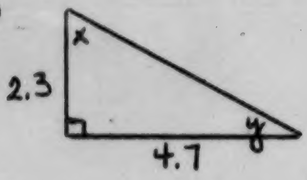
(16) $x = \tan^{-1} 5.14$

(17) $x = \csc^{-1} .37$

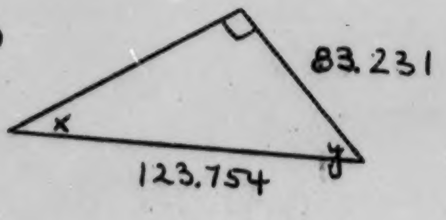
(18) $x = \sec^{-1} .37$

(19 - 22) In each of the following find x and y correct to the nearest tenth.

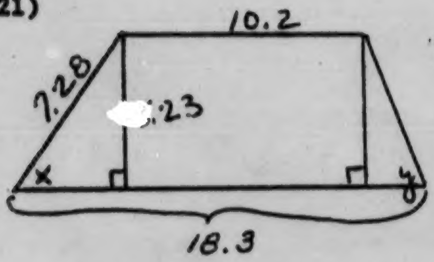
(19)



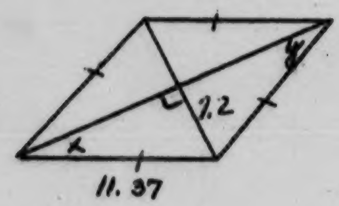
(20)



(21)



(22)



(23) Throughout this section we have repeatedly mentioned the mathematical concept of inverse. Your calculator deals with many operations (functions) that are inverses. Find at least 5 operations and their inverses that are specific calculator keys. Give an example that shows the inverse relationship.

Examples: (a) addition and subtraction $x + a - a = x$

$5 + 7 - 7 = 5$

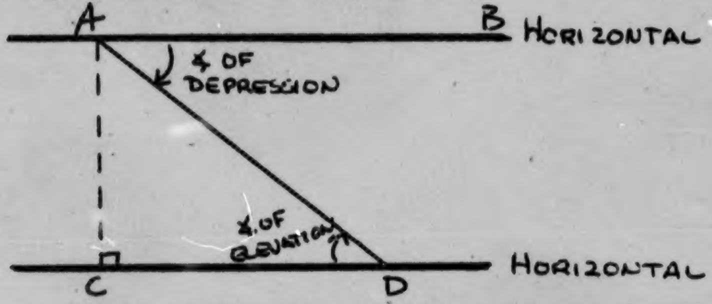
(b) sine x and $\text{sine}^{-1} x$

$\sin^{-1}(\sin 40)$

$\sin^{-1}(.5878)$

3.5 Problem Solving with Right Triangles

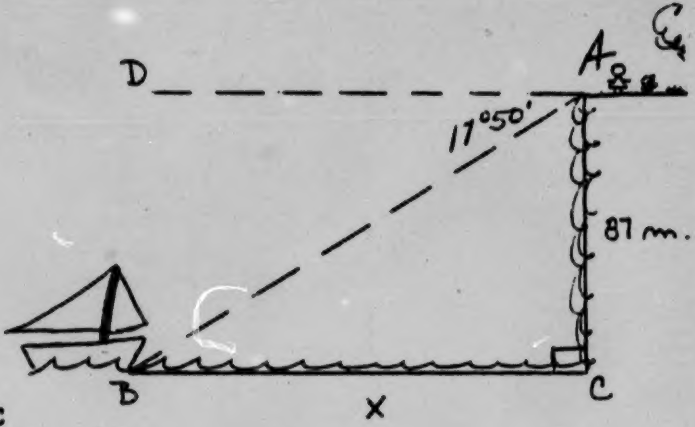
In surveying, angles are measured upward or downward from a horizontal line. The measuring instrument is set up with its line of sight on a horizontal plane. To sight an object higher than the horizontal the line of sight is elevated. To sight an object lower than the horizontal the line of sight is depressed. The angles formed by elevating or depressing the line of sight are called the angle of elevation and the angle of depression respectively.



In the diagram above ∡ BAD is the angle of depression and ∡ ADC is the angle of elevation. Notice that since $\overline{AB} \parallel \overline{CD}$, ∡ BAD ≅ ∡ ADC. Notice also that ∡ ABD is not an angle of △ACD while ∡ ADC is an angle of △ACD.

Example 1:

A person on a cliff 87 meters above a lake measures the angle of depression of a boat to be 17° 50 . To the nearest meter, how far is the boat from the foot of the cliff?



∡ DAB ≅ ∡ ABC

In right △ABC, $\cot 17^{\circ} 50' = \frac{x}{87}$

$$x = 87 \cdot (\cot 17^{\circ} 50')$$

HP-33 solution

(display) 17.50

$\boxed{g}$ $\boxed{\to H}$

(display) 17.8333 [convert degrees - minutes to decimal degrees]

$\boxed{f}$ $\boxed{\tan}$

(display) 0.3217 [.3216 = tan 17.83°]

$\boxed{g}$ $\boxed{\frac{1}{x}}$

(display) 3.1084 [3.1090 = cot 17.83°]

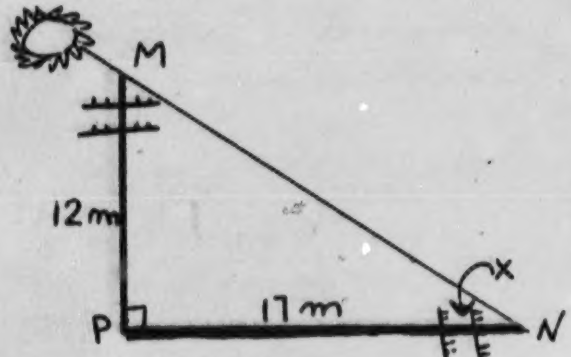
87 $\boxed{\times}$

(display) 270.4326

The boat is 270 meters from the foot of the cliff (correct to the nearest meter).

Example 2:

At a time when a telephone pole 12 meters high casts a shadow 17 meters long find the angle of elevation of the sun, correct to the nearest minute.



In right △ MNP,

$\tan x = \frac{12}{17}$

$x = \tan^{-1}(\frac{12}{17})$

TI 57 solution

12 $\boxed{\div}$ 17 $\boxed{=}$

.7058823529 (display)

$\boxed{\text{INV}}$ $\boxed{\text{2nd}}$ $\boxed{\text{tan}}$

35.21759297 (display)

$\boxed{\text{INV}}$ $\boxed{\text{2nd}}$ $\boxed{\text{D. MS}}$

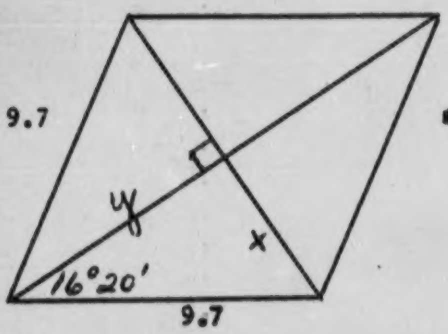
35.13033347 (display)

the angle of elevation of the sun is 35° 13′ (correct to the nearest minute).

The trigonometric solution of right triangles has applications wherever right triangles are found in figures in plane geometry.

Example 3 *

One side of a rhombus is 9.7 inches and one angle 32° 40′. Find the length of each diagonal to the nearest tenth.**



$$16^\circ\ 20' = 16.\overline{33}^\circ$$

$$\sin 16.\overline{33}^\circ = \frac{x}{9.7} \qquad \cos 16.\overline{33}^\circ = \frac{y}{7.7}$$

$$0.2812 = \frac{x}{9.7} \qquad 0.9596 = \frac{y}{9.7}$$

$$2.7279 = x \qquad 9.3085 = y$$

$$5.4558 = 2x \qquad 18.6171 = 2y$$

The diagonals are 5.5 inches and 18.6 inches, correct to the nearest tenth.

---

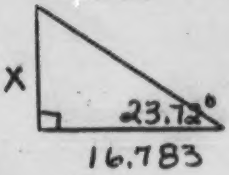
[*] The keystrokes are not provided in this example. Follow the example, using your calculator, to verify each step.

[**] The diagonals of a rhombus are perpendicular, bisect each other and bisect the angles through which they are drawn.

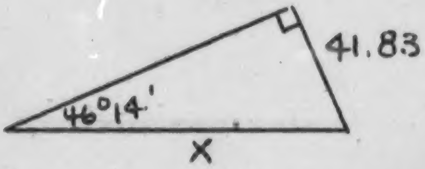
Exercises 3.5

(1 - 10) In each of the following find x correct to the nearest tenth.

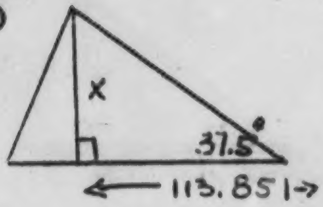
(1)


X
23.72°
16.783

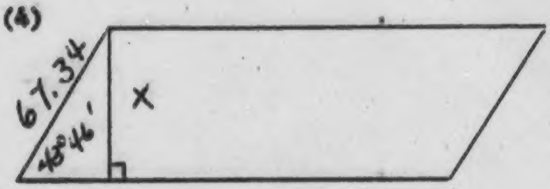
(2)



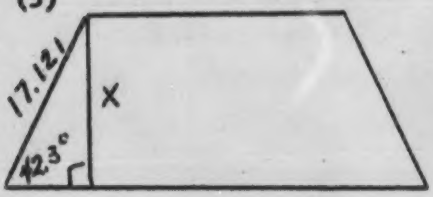
(3)


X
37.5°
113.851

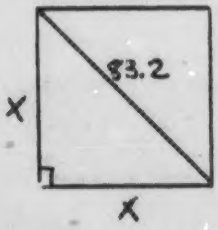
(4)



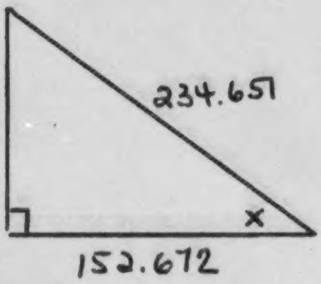
(5)



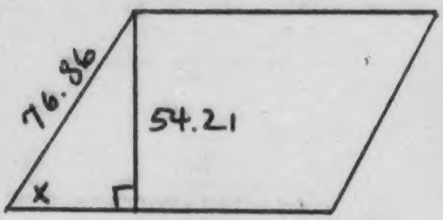
(6)



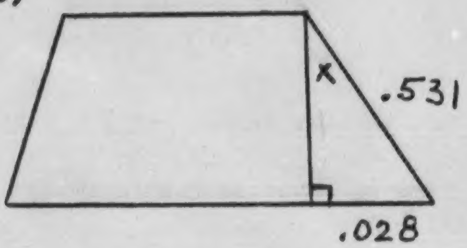
(7)


234.651
X
152.672

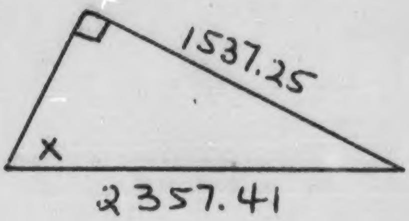
(8)



(9)



(10)


1537.25
X
2357.41

11) A plane takes off from a runway and ascends at an angle of 12.3° with the horizontal. Find to the nearest meter, the altitude of the plane after it has traveled a horizontal distance of 1000 meters.

12) At a point 11.2 meters from the base of a tree, the angle of elevation of the top of the tree is 47° 22′ . Find to the nearest meter the height of the tree.

13) A spotter in a plane at an altitude of 107 meters observes that the angle of depression of a forest fire is 56.7°. How far, to the nearest meter, is the forest fire from the point on the ground directly below the spotter?

14) The lengths of two sides of a parallelogram are 7.2 cm and 11.3 cm. and the measure of angle between them measures 37° 53′. What is the length of the altitude to the longer side?*

15) Find to the nearest decimeter the height of a church spire that casts a shadow of 19.3 meters when the angle of elevation of the sun measures 62.5°.

16) A lighthouse built at sea level is 60 meters high. From its top the angle of depression of a buoy in the ocean measures 18° 45′ . Find the distance from the buoy to the foot of the lighthouse.

17) If the vertex angle of an isosceles triangle measures 63° and each leg 3 inches, find the length of the altitude to the base to the nearest tenth.

18) One diagonal of a rhombus is 28.6 and one side is 15.3. Find the length of the other diagonal and the measure of each angle of the rhombus, to the nearest tenth.

---

*When solving for a measurement, retain the same number of significant decimal figures in the result as were expressed in the original data, unless the problem specifically requests a different accuracy. In this case your answer should be rounded to the nearest tenth.

19) A man on the top of a cliff 350 meters above sea level observes two ships due east of the foot of the cliff. The angles of depression of the two ships measure $18^\circ 30'$ and $32^\circ 15'$. Find the distance between the ships.

20) A vertical tree is growing at the edge of a riverbed. The angle of elevation of the top of the tree from a point directly across the river at the water's edge is $63^\circ 50'$. At another point, 1000 meters from the first point and in line with the first point and the base of the tree, the angle of elevation of the top of the tree is $42^\circ 30'$. Find the width of the river. (Answer: 819 meters)

3.6 Chapter Test

(1 - 10) Decide whether each of the following is true or false.

1) The cosecant of any acute angle is always greater than 1.

2) The tangent of any acute angle is always less than 1.

3) The value of the sine of any acute angle increases as the measure of the angle increases.

4) If θ is an acute angle then $\sin\theta^\circ = \cos(90 - \theta)^\circ$.

5) If θ is an acute angle then $\tan\theta^\circ = \tan(90 - \theta)^\circ$.

6) As the measure of an acute angle increases, the value of its cosine increases.

7) The tangent of any acute angle is always greater than 0 and less than 1.

8) $90^\circ \equiv 100$ grads

9) $15^\circ 3' = 15.50^\circ$

10) $\sin(\arccos 1) = 0$

(11 - 14) Evaluate each of the following expressions

a) algebraically
b) on your calculator. Express your answer to four decimal digits.

11) $\sin 45^\circ + \cos 30^\circ$

12) $\cos^2 45^\circ + \sin^2 45^\circ$

13) $\tan 30^\circ + \cot 30^\circ$

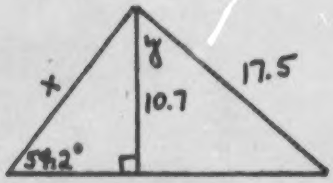
14) $\sec 30^\circ + \csc 60^\circ$

(15 - 16) Convert each of the following to decimal degrees.

15) $37^\circ 15' 20''$

16) 87.25 grads

17) Find x and y correct to the nearest tenth.



(18 - 19) Answer one of the following questions.

18 a) What is the perimeter of a regular polygon of 100 sides inscribed in a circle of radius 1. [Each interior angle of a regular polygon having n sides is $(\frac{(n-2)180}{n})^{\circ}$.

b) Why is your answer near 2π?

19) A plane flying at an altitude of 700 ft. passes directly overhead. Three seconds later its angle of elevation is 23°.

a) Determine its speed in feet per second to the nearest foot per second.

b) Determine its speed in miles per hour.

$$(60 \text{ mph} = \frac{60 \text{ miles}}{1 \text{ hour}} = \frac{60 \text{ miles}}{60 \text{ min.}} = \frac{60 \text{ mi.}}{3600 \text{ sec.}} =$$

$$\frac{5280(60)}{3600 \text{ sec}} = \frac{88 \text{ ft}}{1 \text{ sec}} = 88 \text{ ft/sec.})$$

CHAPTER 4. TRIGONOMETRY BEYOND THE RIGHT TRIANGLE

In this chapter you will become familiar with important aspects of trigonometry that do not specifically involve triangles.

4.1 Extending the Domain of the Trigonometric Functions

So far we have restricted our discussion of trigonometric functions to only acute angles. We now wish to carefully examine the values of trigonometric functions for other types of angles. Complete the following table. Set your calculator to 4 decimal places.

x*	sin x	cos x	tan x
0			
15			
30			
45			
60			
75			
90			
105	0.9659		
120		-0.5000	
135			-1.0000
150			
165		-0.9659	
180			
195			
210	-0.5000		
225			

---

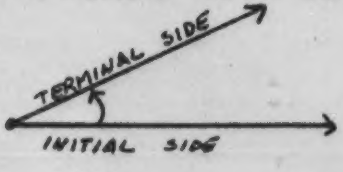
*In this table angles are measured in degrees.

x	sin x	cos x	tan x
240	-0.8660		
255			
270			*
285			
300		0.5000	
315			
330			-0.5774
345			
360		1.0000	

We know that angles having measures of greater than 90° exist. A reasonable question to ask is, "What do these numbers in the table mean and where do they come from?"

In order to answer these questions it will be necessary to redefine the trigonometric functions. This does not mean that the definitions in Chapter 3 are wrong. They are adequate if we restrict our domain to acute angles and right triangles. We wish now to expand our considerations, therefore we need a more appropriate method of dealing with angles and the trigonometric functions associated with them.

An angle is formed by two rays that have a common endpoint. A more dynamic concept of angle involves a movable ray and a fixed ray that have a common vertex. The fixed ray is called the initial side of the angle and the movable ray is called the terminal side of the angle.



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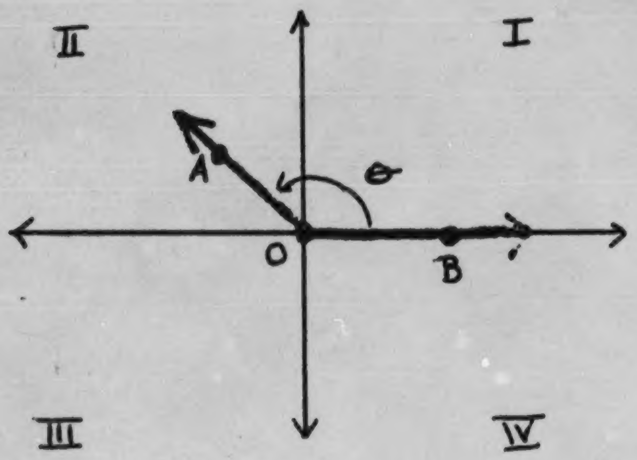
[*] Different calculators give different answers, for example: 9.9999999×10^{99}, $-9.9999999 \times 10^{99}$, error.

When this angle is positioned in the Cartesian plane so that the vertex is at the origin, the initial side along the positive direction of the x-axis and the terminal side somewhere in the plane the angle is considered to be in standard position.



$\measuredangle AOB$ or $\measuredangle\theta$*
is an obtuse angle
in standard position

Acute angles (measures between 0° and 90°) have their terminal sides in quadrant I. Obtuse angles (measures between 90° and 180°) have their terminal sides in quadrant II. Angles whose measures are between 180° and 270° (called reflex angles) have their terminal sides in quadrant III. Angles whose measures are between 270° and 360° (also called reflex angles) have their terminal sides in quadrant IV.

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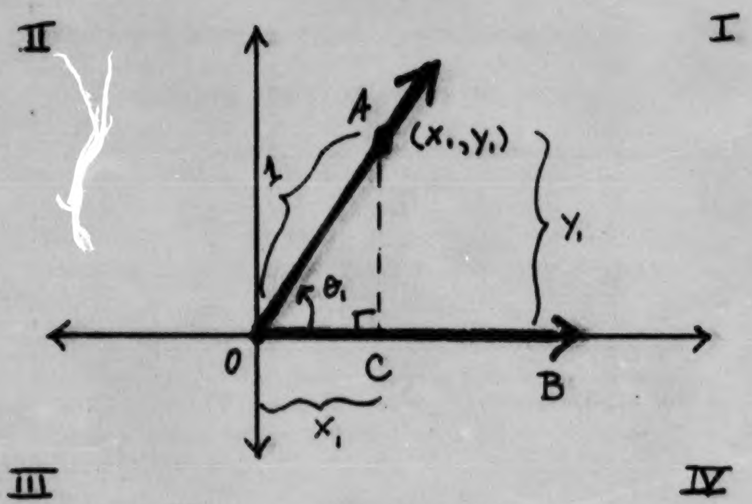
* Don't panic, θ, pronounced "theta" is a Greek letter that is traditionally used as a variable in higher-level mathematics.

We wish to look carefully at an acute angle in standard position.



Let A be a point on the terminal side of $\measuredangle\, \theta_1$ and let (x_1, y_1) be the coordinates of A. Let $\overline{AC} \perp \overline{OB}$, so AC = y and OC = x and $\triangle$ AOC is a right triangle. For convenience we shall assume that AO = 1.[*] Our original definitions of the trigonometric functions yield the following equations:

$$\sin \theta_1 = \frac{\text{ordinate of A}}{\text{radius}} = \frac{AC}{AO} = \frac{y_1}{1} = y_1$$

$$\cos \theta_1 = \frac{\text{abscissa of A}}{\text{radius}} = \frac{OC}{AO} = \frac{x_1}{1} = x_1$$

$$\tan \theta_1 = \frac{\text{ordinate of A}}{\text{abscissa of A}} = \frac{AC}{OC} = \frac{y_1}{x_1}$$

$$\cot \theta_1 = \frac{\text{abscissa of A}}{\text{ordinate of A}} = \frac{OC}{AC} = \frac{x_1}{y_1}$$

$$\sec \theta_1 = \frac{\text{radius}}{\text{abscissa of A}} = \frac{AO}{OC} = \frac{1}{x_1}$$

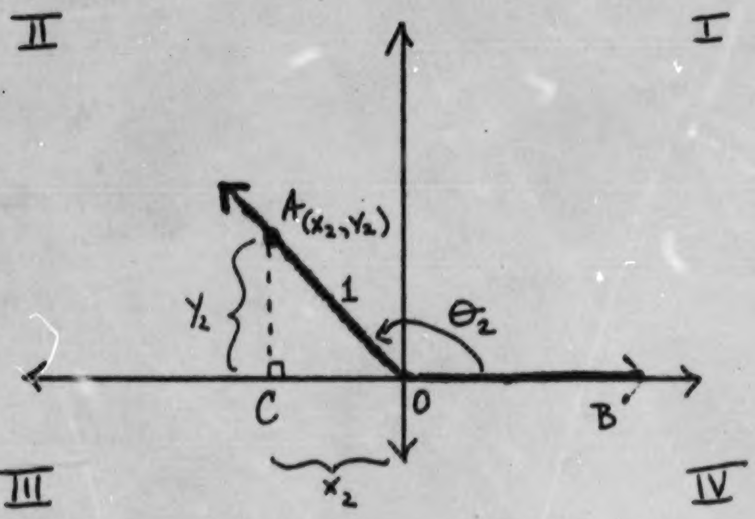
---

[*] Thus A is a point on a circle whose center is the origin and whose radius is 1. This circle is usually referred to as *the unit circle*. We will say more about this special circle in the next section.

$$\csc\theta_1 = \frac{\text{radius}}{\text{ordinate of A}} = \frac{AO}{AC} = \frac{1}{y_1}$$

Notice that OC, AC and OA are all positive lengths so all the values of the trigonometric functions for acute angles are positive.

Now let us look at an obtuse angle, using this same idea.



let (x_2, y_2) be the coordinates for A and let AO = 1.

$$\sin\theta_2 = \frac{\text{ordinate of A}}{\text{radius}} = \frac{AC}{AO} \qquad \frac{y_2}{1} = y_2$$

$$\cos\theta_2 = \frac{\text{abscissa of A}}{\text{radius}} = \frac{CO}{AO} \qquad \frac{x_2}{1} = x_2$$

$$\tan\theta_2 = \frac{\text{ordinate of A}}{\text{abscissa of A}} = \frac{AC}{CO} = \frac{y_2}{x_2}$$

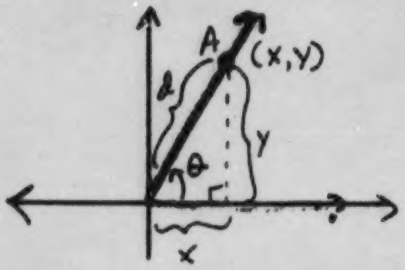
$$\cot\theta_2 = \frac{\text{abscissa of A}}{\text{ordinate of A}} = \frac{CO}{AC} = \frac{x_2}{y_2}$$

$$\sec\theta_2 = \frac{\text{radius}}{\text{abscissa of A}} = \frac{AO}{CO} = \frac{1}{x_2}$$

$$\csc\theta_2 = \frac{\text{radius}}{\text{ordinate of A}} = \frac{AO}{AC} = \frac{1}{y_2}$$

Notice that our new definitions are compatible with our original definitions of the trigonometric functions and they also allow us to extend our domain. Since x_2 represents a negative number $\cos\theta_2 < 0$, $\tan\theta_2 < 0$, $\cot\theta_2 < 0$ and $\sec\theta_2 < 0$. The radius, OA, is always considered positive.

We can now redefine the trigonometric functions. Let A be a point on the terminal side of $\angle\theta$. Let d be the distance from A to the origin.



$$\sin\theta = \frac{\text{ordinate of A}}{\text{distance to origin}} = \frac{y}{d}$$

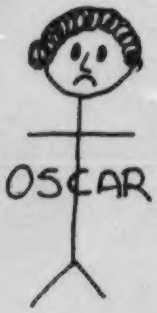
$$\cos\theta = \frac{\text{abscissa of A}}{\text{distance to origin}} = \frac{x}{d}$$

$$\tan\theta = \frac{\text{ordinate of A}}{\text{abscissa of A}} = \frac{y}{x}$$

$$\cot\theta = \frac{\text{abscissa of A}}{\text{ordinate of A}} = \frac{x}{y}$$

$$\sec\theta = \frac{\text{distance to origin}}{\text{abscissa of A}} = \frac{d}{x}$$

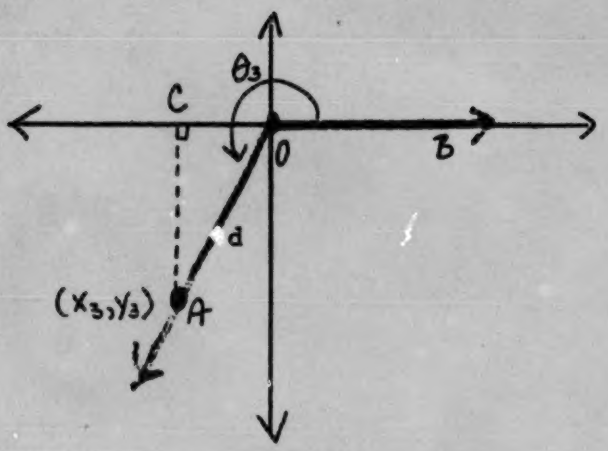
$$\csc\theta = \frac{\text{distance to origin}}{\text{ordinate of A}} = \frac{d}{y}$$



Oscar
didn't
always
do
outstanding
algebra !

Exercises 4.1

(1 - 8) Use the following diagram to characterize reflex angles whose measures are between 180° and 270°. Remember that $OA = +1$.


θ_3
C
0
B
d
(x_3, y_3)
A
1

(1) sin θ_3 =

(2) cos θ_3 =

(3) tan θ_3 =

(4) cot θ_3 =

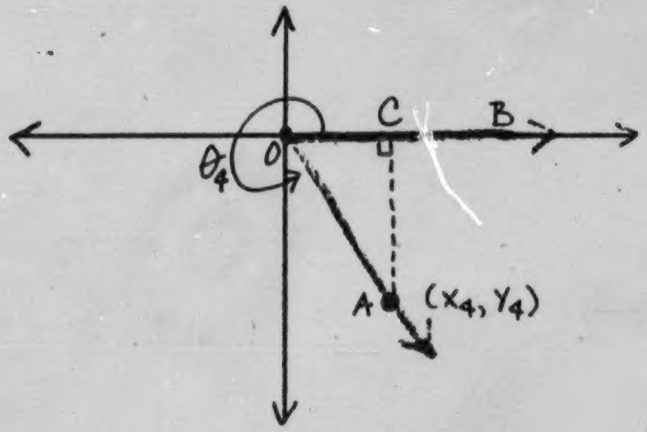
(5) sec θ_3 =

(6) csc θ_3 =

(7) Why is tan $\theta_3 > 0$ and cot $\theta_3 > 0$?

(8) Which trigonometric functions of θ_3 are negative? Which are positive? Why?

(9 - 16) Use the following diagram to characterize reflex angles whose measures are between 270° and 360°. Remember that OA = +1.


C
B
0
θ_4
A
(x_4, y_4)

(9) $\sin \theta_4 =$

(10) $\cos \theta_4 =$

(11) $\tan \theta_4 =$

(12) $\cot \theta_4 =$

(13) $\sec \theta_4 =$

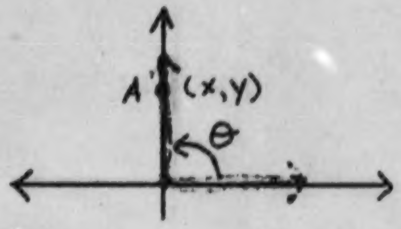
(14) $\csc \theta_4 =$

(15) Why is $\sin \theta_4 < 0$ and $\cos \theta_4 > 0$?

(16) Which trigonometric functions of θ_4 are negative? Which are positive? Why?

(17 - 30) Explain each of the following:

__Example__: $\tan 90^\circ$ is undefined



$x = 0, \quad y = 1$

$$\tan 90^\circ = \frac{\text{ordinate of A}}{\text{abscissa of A}} = \frac{1}{0} \quad \text{undefined}$$

(17) $\sin 0^\circ = 0.0000$

(18) $\cos 0^\circ = 1.0000$

(19) $\sin 90^\circ = 1.0000$

(20) $\cos 90^\circ = 0.0000$

(21) $\sin 180^\circ = 0.0000$

(22) $\cos 180^\circ = -1.0000$

(23) $\sin 270^\circ = -1.0000$

(24) $\cos 270^\circ = 0.0000$

(25) $\sin 360^\circ = 0.0000$

(26) $\cos 360^\circ = 1.0000$

(27) $\tan 0^\circ = 0.000$

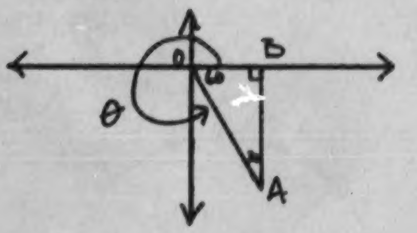
(28) $\tan 180^\circ = 0.0000$

(29) $\tan 270^\circ$ is undefined

(30) $\tan 360^\circ = 0.0000$

(31 - 40) Verify that the following entries in your table on page (4. 1 - 1, 4. 1-2) are correct.

Example: $\sin 300^\circ = -0.8660$



$m \measuredangle AOB = 300^\circ$

$\triangle AOB$ is a 30-60-90 $\triangle$

$OA = 1$ so $OB = .5$

$AB = .5\sqrt{3}$ so A has coordinates $(.5, -.5\sqrt{3})$

$$\sin 300^\circ = \frac{\text{ordinate of A}}{\text{radius}} = \frac{-.5\sqrt{3}}{1} = -.5\sqrt{3} = -.5(1.7321)$$

$$= -0.8660$$

(31) $\sin 45 = 0.7071$

(32) $\cos 135 = -0.7071$

(33) $\tan 240 = 1.7321$

(34) $\cot 330 = -1.7321$

(35) $\sec 300 = 2.0000$

(36) $\sin 60 = .8660$

(37) $\cos 225 = -0.7071$

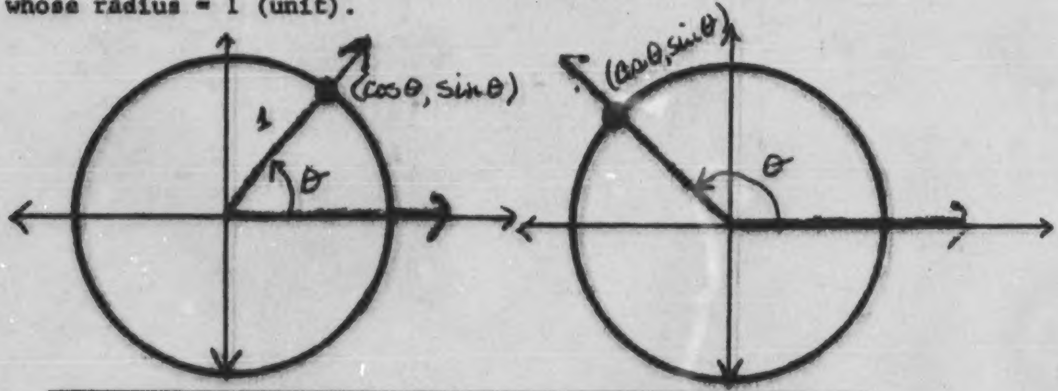
(38) $\csc 210 = -2.0000$

(39) $\tan 150 = -0.5774$

(40) $\tan 135 = -1.0000$

4.2 The Unit Circle

A careful look at the unit circle can help us analyze some of the properties of the trigonometric function. Recall that the unit circle is defined to be the circle whose center is the origin and whose radius = 1 (unit).

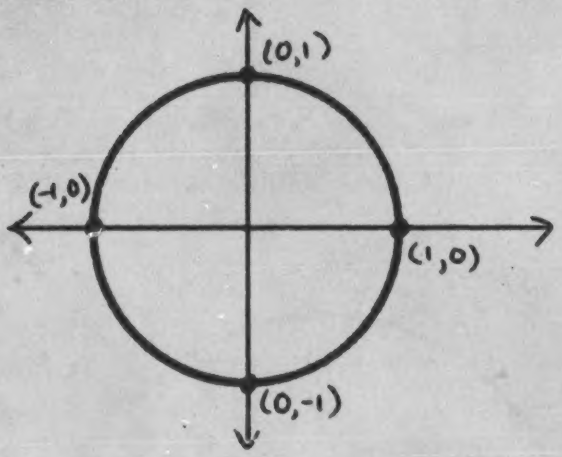


Any point, A, on the unit circle has coordinates ($\cos\theta$, $\sin\theta$) where θ is the measure of the angle whose terminal side passes through A.

We previously verified the identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\text{ordinate}}{\text{abscissa}}$$

Thus, knowing the coordinates of a point, A, on the unit circle is sufficient to describe any trigonometric function of any angle whose terminal side passes through A. Let us consider the intersections of the unit circle with the axes.



Example

(1,0) is on the terminal side of an angle whose measure is 0° or 360°.

Thus $\sin 0^\circ = \sin 360^\circ = 0$

$\cos 0^\circ = \cos 360^\circ = 1$

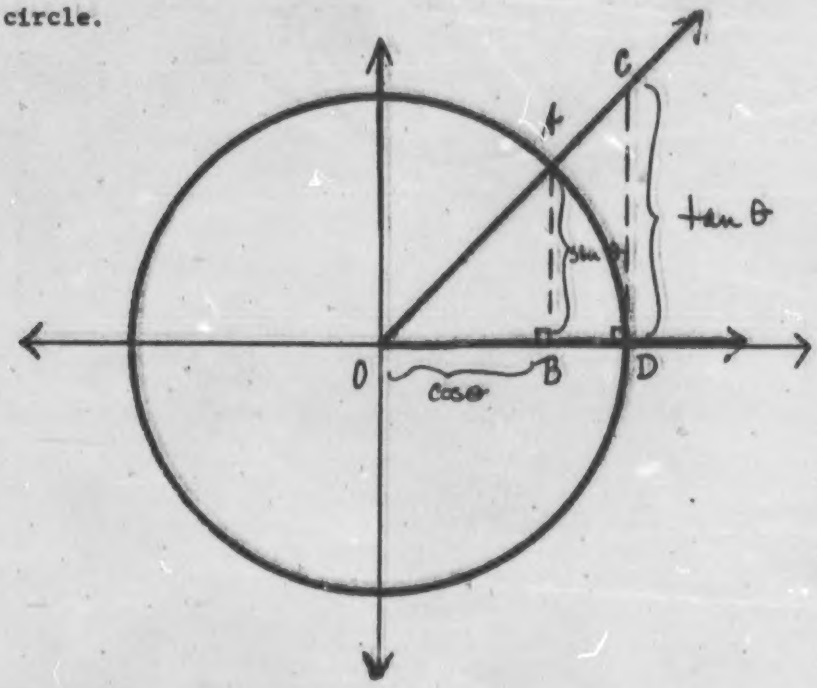
$\tan 0^\circ = \tan 360^\circ = \frac{\sin 360}{\cos 360} = \frac{0}{1} = 0$

$\cot 0^\circ = \cot 360^\circ$ undefined

$\sec 0^\circ = \sec 360^\circ = 1$

$\csc 0^\circ = \csc 360^\circ =$ undefined

By means of a unit circle we can represent the values of the trigonometric functions as line segments associated with the circle.

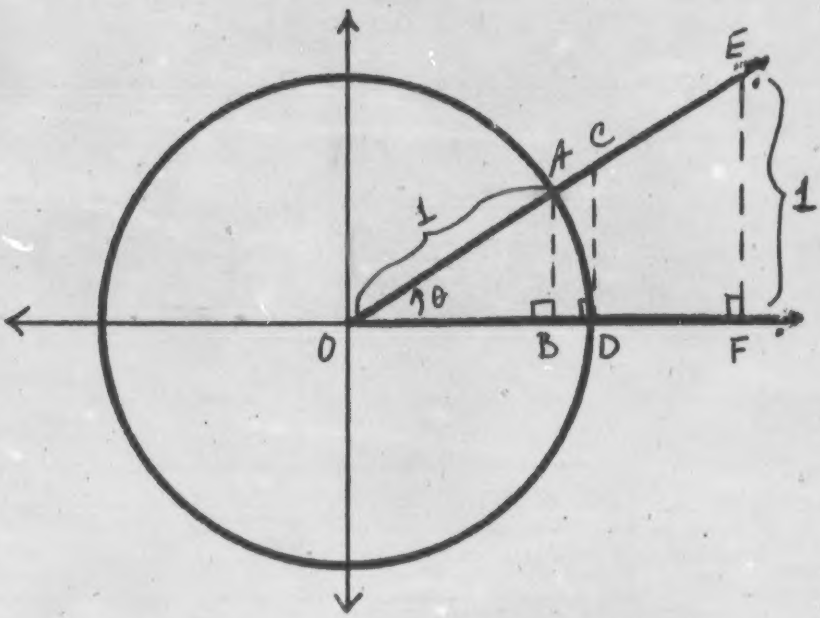


$\sin A = (\cos\theta, \sin\theta)$ then $AB = \sin\theta$

$OB = \cos\theta$

To represent $\tan\theta$ as a line segment we need to convert $\frac{AB}{OB}$ into a ratio whose denominator is 1. Right triangle ABO is similar to right triangle CDO so $\frac{AB}{BO} = \frac{CD}{DO}$. $DO = 1$ thus $\tan\theta = CD$. This may be one reason why this function is named "tangent".

To represent the other functions we will use the same technique.



cot $\theta = \frac{OB}{AB}$. Construct $\triangle$ EFO so that EF = 1 and

$\triangle EFO \sim \triangle ABO$ so $\frac{OB}{AB} = \frac{OF}{EF}$ = OF and cot θ = OF.

sec $\theta = \frac{OA}{OB}$. Construct $\triangle$ CDO so that OD = 1 and

$\triangle CDO \sim \triangle ABO$ so $\frac{OA}{OB} = \frac{OC}{OD}$ = OC and sec θ = OC

csc $\theta = \frac{OA}{AB}$. Since EF = 1 and $\triangle EFO \sim \triangle ABO$

$\frac{OA}{AB} = \frac{OE}{EF}$ = OE and csc θ = OE.

The trigonometric functions can be represented as line segments in the other quadrants by a similar technique. Some adjustments are necessary so that segments can represent negative values. You may wish to consider this case in exercises 28 - 30.

Exercises

(1) Let A = $(\cos\theta, \sin\theta)$ be a point on the unit circle. If θ is an acute $\angle$ then A is in the first quadrant and $\cos\theta$ and $\sin\theta$ are both positive so the other trigonometric functions are also positive. Using this same reasoning, complete the following table.

θ	A	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
acute	1st quadrant	pos.	pos.	pos.	pos.	pos.	pos.
obtuse	2nd quadrant		neg.				
reflex $\angle$ 270	3rd quadrant					neg.	
270 $\angle$ reflex $\angle$ 360	4th quadrant	neg.					

(2) Suggest a reason why the secant function is named "secant".

(3 - 5) Using the Pythagorean relation and the unit circle on p. 4. 2 - 3 and 4. 2 - 4 complete the following equations:

(3) $\sin^2\theta + \cos^2\theta =$ ______ (Hint: $\triangle$ABO)

(4) $1 + \tan^2\theta =$ ______ (Hint: $\triangle$CDO)

(5) $\cot^2\theta + 1 =$ ______

(6 - 11) Determine whether each of the following real numbers are positive, negative or zero and state a reason for your answer.

(6) $\sin 323^\circ$

(7) $\cos 78^\circ$

(8) $\cot 215^\circ$

(9) $\csc 293^\circ$

(10) $\sec 88^\circ$

(11) $\tan 157^\circ$

(12 - 19) For each of the following real numbers:

(a) Use your calculator to find the value and

(b) Suggest a reason why your answer to (a) is reasonable

(12) cot 725° (13) tan 1020°

(14) cos 512° (15) sin 1432°

(16) sin (-115°) (17) cos (-90°)

(18) tan (-200°) (19) csc (-290°)

(20-27) The circumference of a circle = $2\pi r$ where r is the radius of the circle. The unit circle has circumference = $2\pi(1) = 2\pi$. Determine the lengths of the arcs intercepted on the unit circle by each of the following angles. Express your answers (a) in terms of π, (b) to the nearest tenth.

Example:

$$60^\circ = \frac{1}{6}(360)$$

$$\frac{1}{6}(2\pi) = \frac{\pi}{3} = 1.0472 \text{ units}$$

A 60° angle intercepts an arc of $\frac{\pi}{3}$ or 1.0472 units on the unit circle.

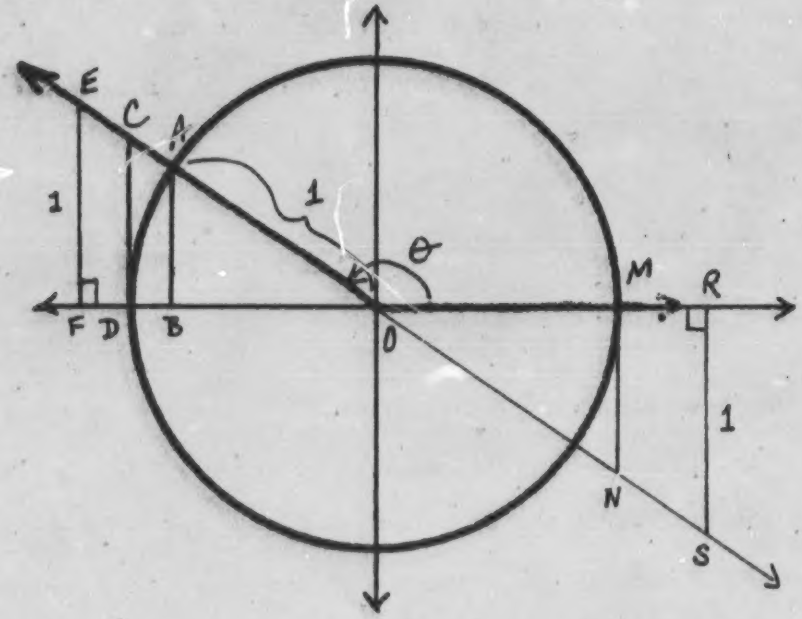
(20) 90° (21) 135°

(22) 180° (23) 225°

(24) 330° (25) 120°

(26) 210° (27) 345°

(28 - 30) Let θ be an obtuse angle.



$\sin\theta = AB$ where AB has positive orientation so $AB > 0$

$\cos\theta = OB$ where OB has negative orientation so $OB < 0$

$\tan\theta = \frac{AB}{OB} = \frac{CD}{OD} = CD$ but $\tan\theta < 0$ when $90 < \theta < 180$.

To remedy this situation construct $\triangle MNO \cong \triangle DCO$. $CD = MN$

so $\tan\theta = MN$ which has negative orientation so $MN < 0$.

(28) Represent $\cot\theta$ as a line segment when $90 < \theta < 180$.

(29) Represent $\sec\theta$ as a line segment when $90 < \theta < 180$.

(30) Represent $\csc\theta$ as a line segment when $90 < \theta < 180$.

(31 - 35) Determine the measure of θ to the nearest tenth. Assume θ is in standard position and

(31) θ has $(-3, -4)$ on its terminal side.

(32) θ has $(-1, \sqrt{3})$ on its terminal side.

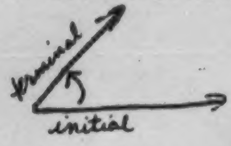
(33) θ has $(7, -10)$ on its terminal side.

(34) θ has $(2, -3)$ on its terminal side.

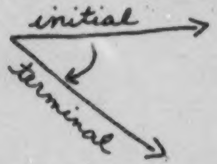
(35) θ has $(1, 5)$ on its terminal side.

4.3 More About Angle Measure.

So far we have been considering angles that have been generated when their terminal sides have been rotated away from their initial sides in a counterclockwise direction.



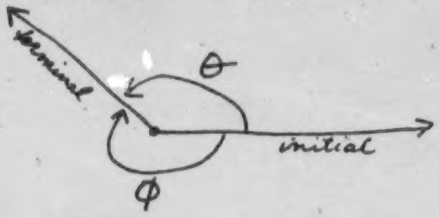
These angles have a positive sense. An angle can be formed by rotating its terminal side clockwise away from its initial side.



These angles have a negative sense.

A negative angle is not an angle that is less than zero, any more than -2 meters is a distance less than zero. If distance north is considered positive then distance south is considered negative. The sign of an angle is similarly a matter of direction.

Every negative angle corresponds to a positive angle.



Thus $\sin\theta = \sin\phi$ * $\quad$ $\cot\theta = \cot\phi$

$\cot\theta = \cos\phi$ $\quad$ $\sec\theta = \sec\phi$

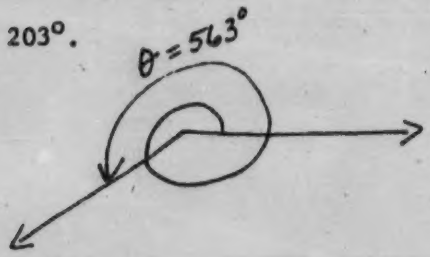
$\tan\theta = \tan\phi$ $\quad$ $\csc\theta = \csc\phi$

Notice that, in general, $-\sin\theta \neq \sin(-\theta)$

---

* ϕ or phi, pronounced "fī" is another Greek letter often used in higher mathematics to represent a variable.

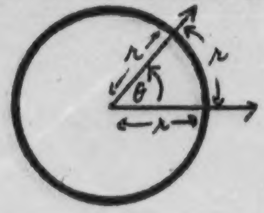
Since the movable ray of an angle can make more than one revolution, the angle that is formed does not need to be less than 360°. Therefore an angle whose measure is 563° is one revolution (360) plus 203°.



The values of the trigonometric function for 563° are the same as for 203° because the initial and terminal sides of both of these angles coincide.

We have been measuring angles by decimal degrees, degrees-minutes-seconds and grads. Another unit, the radian, is often used in higher mathematics, science and engineering.

One radian is the measure of a positive angle which intercepts an arc of length r units on a circle of radius r units.



θ is an angle of 1 radian.

The circumference of a circle is $2\pi r$.

Thus 2π radians = 1 complete rotation = 360°.

$$1 \text{ radian} = \frac{360}{2\pi}^\circ = \frac{180}{\pi}^\circ$$

or

$$\pi \text{ radians} = 180^\circ$$

Later we shall see that radian measure is a convenient way to represent angle measure because it is a "linear" unit of measure.

Most scientific calculators have a radian mode that is not the "wake-up" mode of the calculator.

Example: Find cos $\frac{4\pi}{3}$ radians

HP-33		TI-57	
$\boxed{\text{ON}}$	(Calculators "wake-up")	$\boxed{\text{ON}}$	
$\boxed{\text{g}}$ $\boxed{\text{RAD}}$		$\boxed{\text{2nd}}$ $\boxed{\text{RAD}}$	
0.000	(display)	0	(display)
$\boxed{\text{g}}$ $\boxed{\pi}$		$\boxed{\text{2nd}}$ $\boxed{\pi}$	
3.1416	(display)	3.1415927	(display)
$\boxed{4}$ $\boxed{\times}$		$\boxed{\times}$ $\boxed{4}$ $\boxed{\div}$	
12.5664	(display)	12.566371	(display)
$\boxed{3}$ $\boxed{\div}$		$\boxed{3}$ $\boxed{=}$	
4.1888	(display)	4.1887902	(display)
$\boxed{\text{f}}$ $\boxed{\text{cos}}$		$\boxed{\text{2nd}}$ $\boxed{\text{cos}}$	
-0.5000	(display)	-0.5	(display)

$$\cos \frac{4\pi}{3} \text{ radians} = -0.5$$

$$\left[\begin{array}{l} \frac{4\pi}{3} \text{ radians} = \frac{4\pi}{3}\left(\frac{180}{\pi}\right) = 240^\circ \\ \cos 240^\circ = -0.5 \end{array}\right]$$

Exercises 4.3

(1) Find the number of decimal degrees in one radian.

(2) Find the number of radians in one decimal degree.

(3) Derive a formula to change degrees to radians.

(4) Derive a formula to change radians to degrees.

(5 - 10) Change each of the following radian measures to decimal degrees.*

(5) $\frac{3\pi}{4}$

(6) 1.23

(7) 15π

(8) $-\frac{2\pi}{3}$

(9) $\frac{5}{6}\pi$

(10) -150

(11-20) Change each of the following degree angle measures to radians.

(11) 240°

(12) 127.5°

(13) 45°

(14) 243.75°

(15) 60°

(16) -15.653°

(17) 30°

(18) 543°

(19) 330°

(20) $397^\circ 15'$

(21-30) For each of the following
- (a) evaluate to the nearest hundredth
- (b) give at least one reason why your answer is reasonable.

(21) $\sin (-45)^\circ$

(22) $\cos \frac{3\pi}{2}$

(23) $\tan 548^\circ$

(24) $\sin 58.3$

(25) $\csc (-2.3\pi)$

(26) $\sec \frac{\pi}{4}$

(27) $\sin 6\pi$

(28) $\tan (-93^\circ 35')$

---

*We will adopt the common convention that if no unit of measure is stated, then the unit is the radian.

(29) $\cot \frac{3\pi}{8}$ (30) $\cos 2\pi$

(31-36) Find at least 3 values of θ so that:

(31) $-\sin \theta = \sin (-\theta)$ (32) $-\cos \theta = \cos (-\theta)$

(33) $-\tan \theta = \tan (-\theta)$ (34) $-\cot \theta = \cot (-\theta)$

(35) $2 \sin \theta = \sin 2\theta$ (36) $3 \cos \theta = \cos 3\theta$

4.4 Trigonometric Equations and Principle Values

You are already familiar with several types of equations. We have specifically discussed exponential and logarithmic equations in sections 2.6 and 2.11. You may now use the knowledge you have gained of the trigonometric functions to solve kinds of equations different from those you have studied before. To solve a trigonometric equation you must find a collection of values, within a specific domain, which satisfies the given relationship. The equation $\sin x = .5$ has as roots 30°, 150°, 390°, 510°, ... Generally the domain of the variable is stated in each problem. The values within the specified domain that satisfy the given equation are the elements of the solution set of the trigonometric equation.

Example 1: Solve the following equation for all positive values of x, $0 < x < 360$. Express your answer to the nearest degree.

$$\cos x = -0.8192$$

$$x = \cos^{-1}(-0.8192)$$

HP-33 solution

0.8192 $\boxed{\text{CHS}}$

-0.8192 (display)

$\boxed{\text{g}}$ $\boxed{\cos^{-1}}$

145 (display)

Thus $x = 145^\circ$ is one root of this equation. In general, the principal value of an inverse trigonometric function is:

- the smallest positive (or zero) angle that satisfies the equation (if the angle is positive or zero)
- the smallest negative (or zero) angle that satisfies the equation (if the angle is negative, or zero)

function	range of principal value*
$\sin^{-1}x, \quad 0 \leq x \leq 1$	0 to 90°, $\frac{\pi}{2}$ radians
$\sin^{-1}x, \quad -1 \leq x \leq 0$	0 to -90°, $-\frac{\pi}{2}$ radians
$\cos^{-1}x, \quad 0 \leq x \leq 1$	0 to 90°, $\frac{\pi}{2}$ radians
$\cos^{-1}x, \quad -1 \leq x \leq 0$	90° to 180°, $\frac{\pi}{2}$ to π radians
$\tan^{-1}x, \quad x \geq 0$	0 to 90°, $\frac{\pi}{2}$ radians
$\tan^{-1}x, \quad x \leq 0$	0 to -90°, $\frac{-\pi}{2}$ radians
$\cot^{-1}x, \quad x \geq 0$	0 to 90°, $\frac{\pi}{2}$ radians
$\cot^{-1}x, \quad x \leq 0$	0 to -90°, $\frac{-\pi}{2}$ radians
$\sec^{-1}x, \quad x \geq 1$	0 to 90°, $\frac{\pi}{2}$ radians
$\sec^{-1}x, \quad x \leq -1$	90 to 180°, $\frac{\pi}{2}$ to π radians
$\csc^{-1}x, \quad x \geq 1$	0 to 90°, $\frac{\pi}{2}$ radians
$\csc^{-1}x. \quad x \leq 1$	0 to -90°, $\frac{-\pi}{2}$ radians

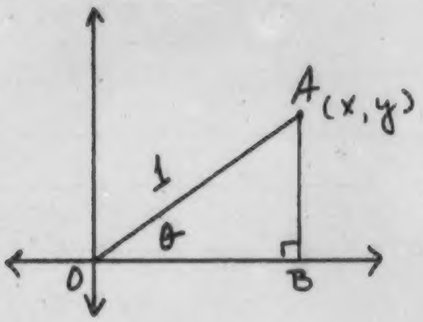
---

* except for values in which division by zero is involved, in which case the value is undefined (e.g., $0^\circ = \csc^{-1}x$).

A closer look at principal values will be presented in section 5.5.

The inverse trigonometric keys (used in conjunction with the $\boxed{\frac{1}{x}}$ key, when necessary) on scientific calculators give these principle values.

Now, we need to determine the other values that are solutions to the equation, $\cos x = -0.8192$. Let us consider acute angles whose terminal sides are in each of the four quadrants and each angle is θ*.



$\sin\theta = \frac{y}{1} = y$

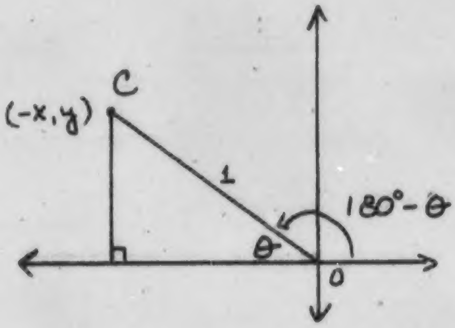
$\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{y}{x}$

$\cot\theta = \frac{x}{y}$

$\sec\theta = \frac{1}{x}$

$\cos\theta = \frac{1}{y}$



$\sin(180^\circ - \theta) = \frac{y}{1} = y$

$\cos(180^\circ - \theta) = \frac{-x}{1} = -x$

$\tan(180^\circ - \theta) = \frac{y}{-x}$

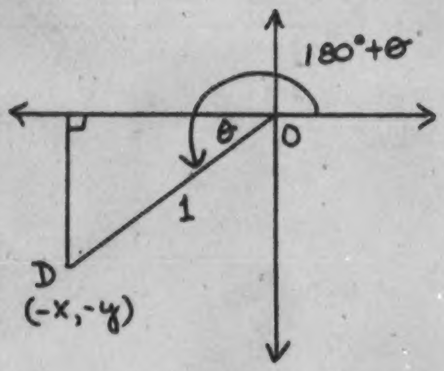
$\cot(180^\circ - \theta) = \frac{-x}{y}$

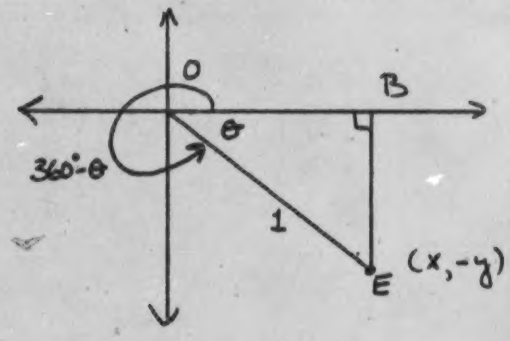
$\sec(180^\circ - \theta) = \frac{1}{-x}$

$\csc(180^\circ - \theta) = \frac{1}{y}$

---

* θ is called the reference angle.





$\sin(180^\circ + \theta) = \frac{-y}{1} = -y$

$\cos(180^\circ + \theta) = \frac{-x}{1} = -x$

$\tan(180^\circ + \theta) = \frac{-y}{-x} = \frac{y}{x}$

$\cot(180^\circ + \theta) = \frac{-x}{-y} = \frac{x}{y}$

$\sec(180^\circ + \theta) = \frac{1}{-x}$

$\csc(180^\circ + \theta) = \frac{1}{-y}$

$\sin(360^\circ - \theta) = \frac{-y}{1} = -y$

$\cos(360^\circ - \theta) = \frac{x}{1} = x$

$\tan(360^\circ - \theta) = \frac{-y}{x}$

$\cot(360^\circ - \theta) = \frac{x}{y}$

$\sec(360^\circ - \theta) = \frac{1}{x}$

$\csc(360^\circ - \theta) = \frac{1}{-y}$

Look carefully at each of these lists of ratios. Notice that:

$\sin(180^\circ - \theta) = \sin\theta$

$\cos(180^\circ - \theta) = -\cos\theta$

$\tan(180^\circ - \theta) = -\tan\theta$

$\cot(180^\circ - \theta) = -\cot\theta$

$\sec(180^\circ - \theta) =- \sec\theta$

$\csc(180^\circ - \theta) = \csc\theta$

$\sin(180^\circ + \theta) = -\sin\theta$

$\cos(180^\circ + \theta) = -\cos\theta$

$\tan(180^\circ + \theta) = \tan\theta$

$\cot(180^\circ + \theta)\ \cot\theta$

$\sec(180^\circ + \theta) = -\sec\theta$

$\csc(180^\circ + \theta) = -\csc\theta$

$\sin(360^\circ - \theta) = -\sin$

$\cos(360^\circ - \theta) = \cos$

$\tan(360^\circ - \theta) = -\tan$

$\cot(360^\circ - \theta) = -\cot$

$\sec(360^\circ - \theta) = \sec$

$\csc(360^\circ - \theta) = -\csc$

this can be summarized as follows:

$$\left| f\begin{pmatrix}180^\circ - \theta \\ 180^\circ + \theta \\ 360^\circ - \theta\end{pmatrix}\right| = f(\theta) \qquad 0^\circ < \theta < 90^\circ$$

f is the same trigonometric function on each side of the equation and the sign of the function is determined by its sign in that quadrant.

Now let us look back at our original example. The cosine function is negative in the second and third quadrants so another solution exists in quadrant III.

$$\cos 145^\circ = \cos(180 - 35)^\circ = -\cos 35^\circ = \cos(180 + 35)^\circ$$
$$= \cos 215^\circ$$

The solution set is $\{145^\circ, 215^\circ\}$.

Example 2: Solve the following equation for values of ϕ between 0° and 360°. Express your answer to the nearest hundredth in decimal degrees.

$$3 \sin\phi + 2 = 8 \sin\phi$$
$$2 = 5 \sin\phi$$
$$.4 = \sin\phi$$
$$\sin^{-1}(.4) = \phi$$
$$23.58^\circ = \phi \qquad \text{(principal value)}$$

Another solution exists in quadrant II where the sine function is also positive.

$$\sin \theta = \sin (180^\circ - \theta)$$

so $$\sin 23.58^\circ = \sin (180^\circ - 23.58^\circ) = \sin 156.42^\circ$$

the solution set is $\{23.58^\circ, 156.42^\circ\}$.

Example 3: Find positive values of θ between 0 and 2π so that $2 \tan \theta + \sqrt{3} = 0$. Express your answers to the nearest tenth of a radian and check.

$$2 \tan \theta + \sqrt{3} = 0$$

$$2 \tan \theta = -\sqrt{3}$$

$$\tan \theta = \frac{-\sqrt{3}}{2}$$

$$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$

(principal value) $$\theta \doteq -40.89^\circ$$

or $$\theta \doteq 319.11^\circ$$

Another solution exists in quadrant II where the tangent function is also negative.

$$\tan (360 - \theta)^\circ = -\tan \theta$$

$$\tan 319.11^\circ = -\tan 40.89^\circ$$

$$-\tan \theta = \tan (180^\circ - \theta)$$

$$-\tan 40.89^\circ = \tan 139.11^\circ$$

$$319.11^\circ = 319.11 \left(\frac{\pi}{180}\right) = 5.6 \text{ (radians)}$$

$$139.11^\circ = 139.11 \left(\frac{\pi}{180}\right) = 2.4 \text{ (radians)}$$

the solution set is $\{5.6, 2.4\}$.

Check: (Remember to set your calculator to radian mode)

$2 \tan 5.6 + \sqrt{3}$	0	$2 \tan 2.4 + \sqrt{3}$	0
$2(-0.8139) + \sqrt{3}$		$2(-0.9160) + \sqrt{3}$	
$-1.6279 + 1.7321$		$-1.8320 + 1.7321$	
0.1042		-0.1000	

Notice the rounding error.

Example 4: Solve for all values x if $0 \leq x \leq 360$. Express your answer to the nearest hundredth. Check your answers.

$$2 = \sqrt{8 \sin x - 1}$$

$$4 = 8 \sin x - 1$$

$$5 = 8 \sin x$$

$$\frac{5}{8} = \sin x$$

$$\arcsin\left(\frac{5}{8}\right) = x$$

$$38.68^{\circ} = x \quad \text{(principal value)}$$

Another solution exists in the second quadrant where the sine function is also positive.

$$\sin x = \sin(180^{\circ} - x)$$

$$\sin 38.68^{\circ} = \sin(180 - 38.68)^{\circ} = \sin 141.32^{\circ}$$

The solution set is $\{38.68^{\circ}, 141.32^{\circ}\}$.

Check

2	$\sqrt{8 \sin 38.68 - 1}$	2	$\sqrt{8 \sin 141.32 - 1}$
2	$\sqrt{8(.62) - 1}$		$\sqrt{8(.62) - 1}$
	$\sqrt{5 - 1}$		$\sqrt{5 - 1}$
	$\sqrt{4}$		$\sqrt{4}$
	2		2

Exercises 4.4

(1 - 10) In each of the following, find the values for $0^\circ \leq \theta \leq 360^\circ$ to the nearest hundredth decimal degree. Check your answers.

(1) $3 \cot \theta = 3\sqrt{3}$

(2) $\cos \theta + \frac{\sqrt{3}}{2} = 0$

(3) $2 \cos \theta + 7 = 0$

(4) $\sqrt{5}(\sin \theta + 1) = 4$

(5) $8.6 \sin \theta = 1 - \sin \theta$

(6) $5 \cos \theta + 6 = 7$

(7) $4 - (2 \cot \theta - .9) - 3 \cot \theta = 1$

(8) $\frac{\tan \theta + 2}{8.1} = \frac{\tan \theta - 2}{3.5}$

(9) $3(\tan \theta - 5.6) = \tan \theta$

(10) $\frac{1 - \cos \theta}{7.1} = \cos \theta$

(11 - 14) Solve for all values of ϕ if $0 \leq \phi \leq 2\pi$. Express your answer correct to the nearest hundredth of a radian. Check your answers.

(11) $2 \tan \phi + .57 = 1.23$

(12) $\sin \phi + 1.8 = \sqrt{5}$

(13) $\sin \phi = \cos\left(\frac{\pi}{2} - \phi\right)$

(14) $\cos \phi - 1.8 = \sqrt{7}$

(15 - 22) Solve for all values of x if $0^\circ \leq x \leq 360^\circ$. Express your answer to the nearest hundredth. Check your answers.

(15) $\sqrt{\tan x + 3} - 1 = 0$

(16) $\sqrt[3]{2 - 2 \cos x} - 3 = 0$

(17) $\sqrt{1 + \cos x} = 3\sqrt{\cos x}$

(18) $\sqrt{1 - \sin x} = \frac{1}{\sqrt{5}}$

(19) $\sqrt[3]{\tan x + 5} = 1$

(20) $\sin^2 x = .837$

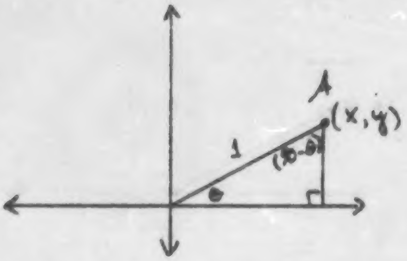
(21) $1 - \tan^2 x = \sqrt{3}$

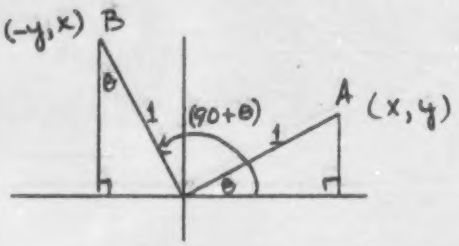
(22) $2.47 = 3 \cos^2 x$*

---

* Look carefully at your solution set to be sure that you have all possible positive angles.

It is possible to relate angles to 90° and 270° rather than 180 and 360. Consider the following diagrams and derive the formulas (identities) by completing the following equations.


A
(x,y)
1
$(90-\theta)$
θ


$(-y,x)$ B
θ
1
$(90+\theta)$
A (x,y)
1
θ

$\sin\theta =$

$\cos\theta =$

$\tan\theta =$

$\cot\theta =$

$\sec\theta =$

$\csc\theta =$

$\sin(90-\theta)^\circ =$

$\cos(90-\theta)^\circ =$

$\tan(90-\theta)^\circ =$

$\cot(90-\theta)^\circ =$

$\sec(90-\theta)^\circ =$

$\csc(90-\theta)^\circ =$

$\sin(90+\theta)^\circ =$

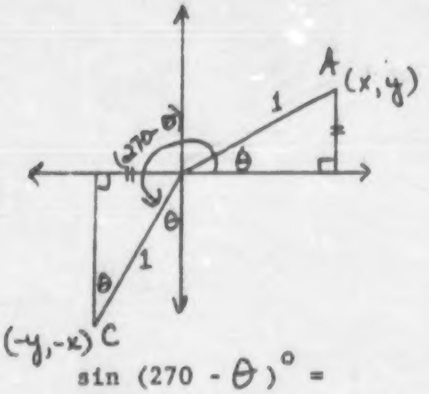
$\cos(90+\theta)^\circ =$

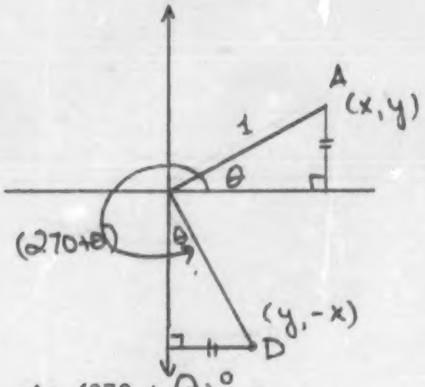
$\tan(90+\theta)^\circ =$

$\cot(90+\theta)^\circ =$

$\sec(90+\theta)^\circ =$

$\csc(90+\theta)^\circ =$


A (x,y)
1
$(270-\theta)$
θ
θ
1
θ
$(-y,-x)$ C


A (x,y)
1
θ
$(270+\theta)$
θ
$(y,-x)$
D

$\sin(270-\theta)^\circ =$

$\cos(270-\theta)^\circ =$

$\sin(270+\theta)^\circ =$

$\cos(270+\theta)^\circ =$

$\tan (270 - \theta)^\circ =$

$\cot (270 - \theta)^\circ =$

$\sec (270 - \theta)^\circ =$

$\csc (270 - \theta)^\circ =$

$\tan (270 + \theta)^\circ =$

$\cot (270 + \theta)^\circ =$

$\sec (270 + \theta)^\circ =$

$\csc (270 + \theta)^\circ =$

(23 - 26) Develop a formula relating the trigonometric functions of the following where $0^\circ < \theta < 90^\circ$

(23) θ and $90^\circ - \theta$

(24) θ and $90^\circ + \theta$

(25) θ and $270^\circ - \theta$

(26) θ and $270^\circ + \theta$

CHAPTER 4 Test

1) Express cos (-250°) as a function of a positive acute angle.

2) In a circle of radius 3, a central angle of 2/3 radian is drawn. Find the length of the arc intercepted by that central angle.

3) Express $\cos(270^\circ - x)$ in terms of $\sin x$.

4) Find in degrees the value of the positive acute angle θ which satisfies the equation $2\cos^2\theta - 1 = 0$.

5) Find the principal value of arc sin (-1). Express answer in terms of π.

6) Express $\frac{7\pi}{6}$ radians in degrees.

7) If $\tan A = -1$ and if A is an angle greater than 0° and less than 180°, find the number of degrees in A.

8) Find the value of $\cos\theta$ if $\sin\theta = \frac{-3}{5}$ and $\tan\theta$ is negative.

9) Express in radians an angle of 144°. [Leave answer in terms of π].

10) If $\cos\theta = \frac{7}{25}$ and θ is in the 4th quadrant, what is the numerical value of $\sin\theta$?

11) The $\cos\left[\frac{\pi}{4} + x\right]$ is equal to

(1) $\frac{1}{2}(\cos x - \sin x)$

(2) $\frac{1}{2}(\cos x + \sin x)$

(3) $\frac{\sqrt{2}}{2}(\cos x - \sin x)$

(4) $\frac{\sqrt{2}}{2}(\cos x + \sin x)$

12) If $\cos\theta = \frac{1}{8}$, then the positive value of $\sin\frac{\theta}{2}$ is

(1) $\frac{3}{2}$ (2) $\frac{\sqrt{7}}{4}$ (3) $\frac{9}{16}$ (4) $\frac{3}{4}$

13) If A and B are positive acute angles and if sin A = $\frac{3}{5}$ and sin B = $\frac{4}{5}$ then sin (A + B) is equal to

(1) 1 (2) 0 (3) $\frac{7}{5}$ (4) $\frac{24}{25}$

14) A value of x for which tan $(x + 20^\circ)$ is undefined is

(1) -20° (2) 70° (3) 160° (4) 340°

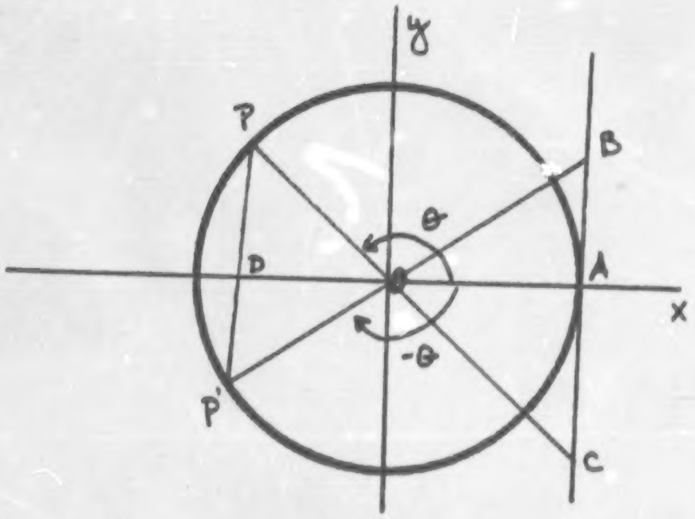
15) If placed in standard position, an angle of $\frac{11\pi}{6}$ radians has the same terminal side as an angle of

(1) -150° (2) -30° (3) 150° (4) 240°

16) In the diagram, trigonometric functions of θ and of $-\theta$ are represented by line segments. Angle AOP = θ and angle AOP' = $-\theta$. The equation of the circle is $x^2 + y^2 = 1$, and the equation of the BC is $x = 1$. Line PDP' is perpendicular to the x-axis at D. From the diagram, select a line segment which represents each of the following:

(a) $\sin\theta$ (b) $\cos\theta$ (c) $\tan\theta$

(d) $\cos(-\theta)$ (e) $\tan(-\theta)$



CHAPTER 5. GRAPHS OF THE TRIGONOMETRIC FUNCTIONS AND THEIR INVERSES.

In this chapter we shall carefully analyze the graphs of the six trigonometric functions and their inverses. We will study the variations of these functions under a variety of conditions.

5.1 Graphs of the sine and cosine functions

In the exercises for this section you will be asked to construct graphs for the cosine functions. This is easily accomplished by making a table of values (by using your calculator) and plotting the resulting points.

When graphing the trigonometric functions we must be careful about the scale that we use on each axis. The graph of any function is a set of points in the coordinate plane located by ordered pairs of real numbers. In the trigonometric functions, although the second element is a real number the first element is usually thought of as being the measure of an angle. If the angles are expressed in radians then we can use the same scale for labeling both axes and we will get the characteristic shapes of the functions.

Example 1: Sketch the graph of $y = \sin x$ for values of x between 0 and 360 (2π) at intervals of 10° ($\frac{\pi}{18}$).

x(degrees)	0	10	20	30	40	50
x(radians	0	$\frac{\pi}{18}$ 0.17	$\frac{\pi}{9}$ 0.35	$\frac{\pi}{6}$ 0.52	$\frac{2\pi}{9}$ 0.70	$\frac{5\pi}{18}$ 0.87
sin x	0	0.17	0.34	0.50	0.64	0.77

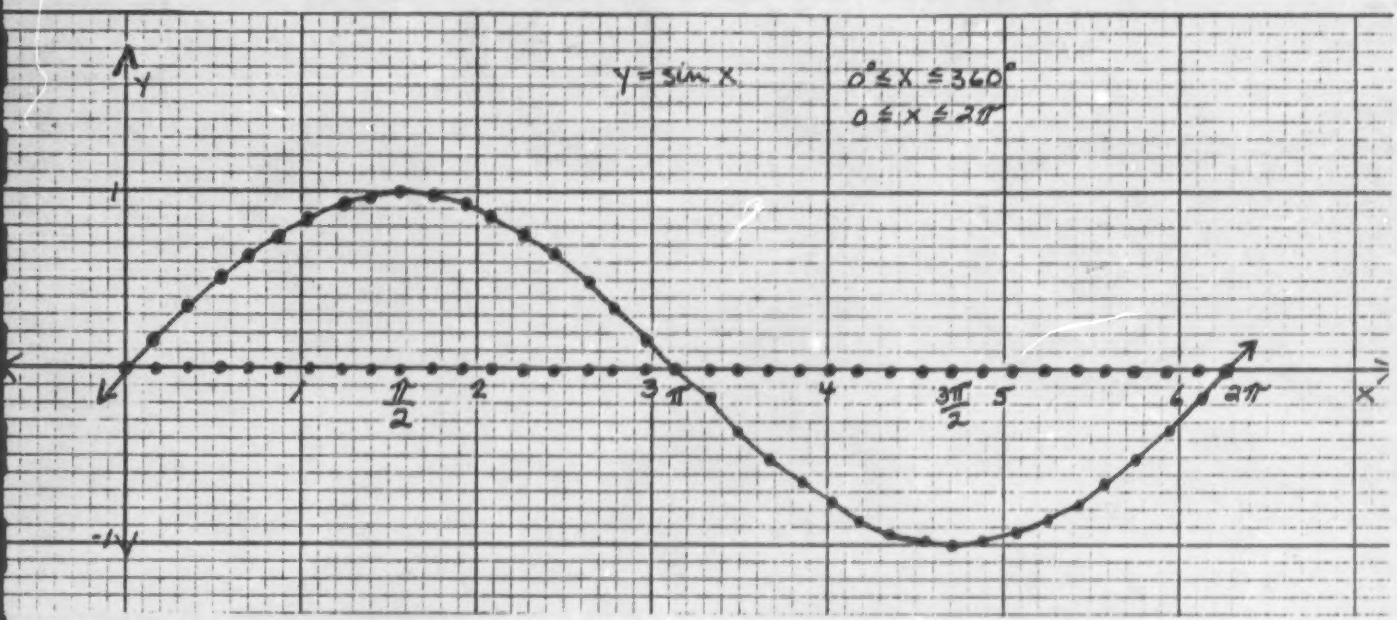
x (degrees)	60	70	80	90	100	110
x (radians)	$\frac{\pi}{3}$ 1.05	$\frac{7\pi}{18}$ 1.22	$\frac{4\pi}{9}$ 1.40	$\frac{\pi}{2}$ 1.57	$\frac{5\pi}{9}$ 1.75	$\frac{11\pi}{18}$ 1.92
sin x	0.87	0.94	0.98	1.00	0.98	0.94

x (degrees)	120	130	140	150	160	170	180
x (radians)	$\frac{2\pi}{3}$ 2.09	$\frac{13\pi}{18}$ 2.27	$\frac{7\pi}{9}$ 2.44	$\frac{5\pi}{6}$ 2.62	$\frac{8\pi}{9}$ 2.79	$\frac{17\pi}{18}$ 2.97	π 3.14
sin x	0.87	0.77	0.64	0.50	0.34	0.17	0

x (degrees)	190	200	210	220	230	240
x (radians)	$\frac{19\pi}{18}$ 3.32	$\frac{10\pi}{9}$ 3.49	$\frac{7\pi}{6}$ 3.67	$\frac{11\pi}{9}$ 3.84	$\frac{23\pi}{18}$ 4.01	$\frac{4\pi}{3}$ 4.19
sin x	-0.17	-0.34	-0.50	-0.64	-0.77	-0.87

x (degrees)	250	260	270	280	290	300
x (radians)	$\frac{25\pi}{18}$ 4.36	$\frac{13\pi}{9}$ 4.54	$\frac{3\pi}{2}$ 4.71	$\frac{14\pi}{9}$ 4.89	$\frac{29\pi}{18}$ 5.06	$\frac{5\pi}{3}$ 5.24
sin x	-0.94	-0.98	-1.00	-0.98	-0.94	-0.87

x (degrees)	310	320	330	340	350	360
x (radians)	$\frac{31\pi}{18}$ 5.41	$\frac{16\pi}{9}$ 5.59	$\frac{11\pi}{6}$ 5.76	$\frac{17\pi}{9}$ 5.93	$\frac{35\pi}{18}$ 6.11	2π 6.28
sin x	-0.77	-0.64	-0.50	-0.34	-0.17	0



One way to determine the values in the table is to write a program. Two programs that can be used are: a) programs to express radians as decimal units

HP-33*

```
PRGM

01   f FIX 2
02   g π
03   1
04   8
05   ÷
06   R/S
07   ENTER
08   g π
09   1
10   8
11   ÷
12   +
13   R/S
14   GTO 07
     RUN
     GTO 00
     R/S
```

TI-57*

```
LRN

00   2nd FIX 2
01   2nd π
02   ÷
03   1
04   8
05   =
06   STO 1
07   R/S
08   2nd Lbl 1
09   +
10   RCL 1
11   =
12   R/S
13   GTO 1
     LRN
     RST
     R/S
```

---

* Notice that these program use different techniques to convert radians to decimals.

b) programs to determine values of sin x

HP-33[*]

```
PRGM
01    f FIX 2
02    ENTER
03    STO 1
04    f sin
05    R/S
06    RCL 1
07    1
08    0
09    +
10    STO 1
11    f sin
12    R/S
13    GTO 06
  RUN
      GTO 00
  R/S
```

TI-57[*]

```
LRN
00    2nd FIX 2
01    STO 1
02    2nd sin
03    R/S
04    2nd Lbl 1
05    RCL 1
06    +
07    1
08    0
09    =
10    STO 1
11    2nd sin
12    R/S
13    GTO 1
  LRN
  RST
  R/S
```

Example 2: Sketch the graph of y = sin x for values of x between -360° (-2π) and 360° (2π) at intervals of 30° ($\frac{\pi}{6}$).

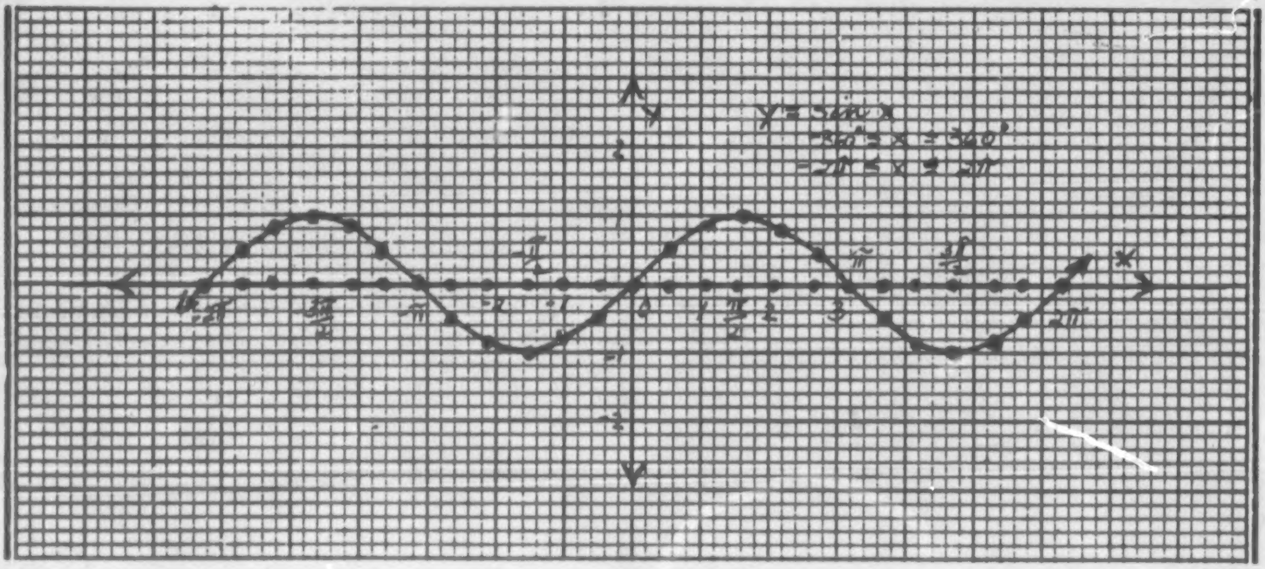
x (degrees)	-360	-330	-300	-270	-240	-210	-180	-150	-120	-90	-60
x (radians)	-2π -6.28	$-\frac{11\pi}{6}$ -5.76	$-\frac{5\pi}{3}$ -5.24	$-\frac{3\pi}{2}$ -4.71	$-\frac{4\pi}{3}$ -4.19	$-\frac{7\pi}{6}$ -3.67	$-\pi$ -3.14	$-\frac{5\pi}{6}$ -2.62	$-\frac{2\pi}{3}$ -2.09	$-\frac{\pi}{2}$ -1.57	$-\frac{\pi}{3}$ -1.05
sin x	0	.5	.87	1	.87	.5	0	-.5	-.87	-1	-.87

---

[*] These two programs use the same technique to determine the values of sin x.

-30	0	30	60	90	120	150
$-\frac{\pi}{6}$ -.52	0	$\frac{\pi}{6}$ .52	$\frac{\pi}{3}$ 1.05	$\frac{\pi}{2}$ 1.57	$\frac{2\pi}{3}$ 2.09	$\frac{5\pi}{6}$ 2.62
-.5	0	.5	.87	1	.87	.5

180	210	240	270	300	330	360
π 3.14	$\frac{7\pi}{6}$ 3.67	$\frac{4\pi}{3}$ 4.19	$\frac{3\pi}{2}$ 4.71	$\frac{5\pi}{3}$ 5.24	$\frac{11\pi}{6}$ 5.76	2π 6.28
0	-.5	-.87	-1	-.87	-.5	0



The programs for example 1 can be easily modified to be appropriate for this problem.

We have drawn these graphs in great detail. If we look carefully they reveal the characteristic properties of the sine function. Notice that y = sin x:

1. has a characteristic shape.
2. is continuous.

3. has all real numbers as its domain.
4. has the set of real numbers from -1 to +1 inclusive as its range.
5. increases in quadrants I and IV.
6. decreases in quadrants II and III.
7. is positive in quadrants I and II.
8. is negative in quadrants III and IV.
9. repeats itself (has a cycle) every 360°.

Several of the types of properties listed above have special names. Properties 5, 6, 7 and 8 are generally referred to as variation, or how the values of y change when the values of x change. Property 4 is called the amplitude or the biggest distance the graph gets from the x-axis. The amplitude of $y = \sin x$ is 1. Property 9 is called the period or the smallest value that when added to any x will give the same y-value. The period of $y = \sin x$ is 360° or 2π. This means that $\sin (x + 360)^\circ = \sin x$.

Exercises 5.1

(1 - 3) Sketch a graph of the cosine function

(a) as x varies from 0 to 2π at intervals of $\frac{\pi}{18}$.

(b) as x varies from -2π to 2π at intervals of $\frac{\pi}{6}$.

1) What is the variation of $y = \text{cosine } x$?

2) What is the amplitude of $y = \text{cosine } x$?

3) What is the period of $y = \text{cosine } x$?

(4 - 7) Use the graph of $y = \sin x$ and $y = \cos x$ to determine the quadrant that the terminal side of angle x would be in if, x is increasing and . . .

4) $\sin x$ is positive and decreasing.

5) $\cos x$ is positive and increasing.

6) $\cos x$ is negative and decreasing.

7) $\sin x$ is negative and decreasing.

(8 - 10) Use the graph of $y = \sin x$ and $y = \cos x$ to determine the positive value of x, less than 2π when:

8) $\sin x = \cos x$, and both are negative.

9) $\sin x$ is negative and $\cos x = 0$.

10) $\sin x = \cos x$, and both are positive.

11) What characteristics do the functions $y = \sin x$ and $y = \cos x$ have in common?

12) What characteristics do the functions $y = \sin x$ and $y = \cos x$ not have in common?

13) Consider the function $y = \cos x + \sin x$.

(a) Without calculating specific values, sketch what you think this graph will look like.

(b) Sketch this graph for values of x between 0 and 2π at intervals of $\frac{\pi}{12}$.

14) Consider the function $y = \cos x - \sin x$

(a) Without calculating specific values, sketch this graph.

(b) Sketch this graph for values of x between 0 and 2π at intervals of $\frac{\pi}{12}$.

15) How do you think the graph of $y = \sin x - \cos x$ would look different from the graph of $y = \cos x - \sin x$? Verify your answer.

5.2 Graphs of the tangent function.

The function y = tan x can be graphed in the same manner as y = sin x and y = cos x.

Example 1: Sketch the graph of y = tan x for values of x between 0 and 360 (2π) at intervals of 10° ($\frac{\pi}{18}$).

x degrees	0	10	20	30	40	50
x radians	0	$\frac{\pi}{18}$ (0.17)	$\frac{\pi}{9}$ (0.35)	$\frac{\pi}{6}$ (0.52)	$\frac{2\pi}{9}$ (0.70)	$\frac{5\pi}{18}$ (0.87)
tan x	0	0.18	0.36	0.58	0.84	1.19

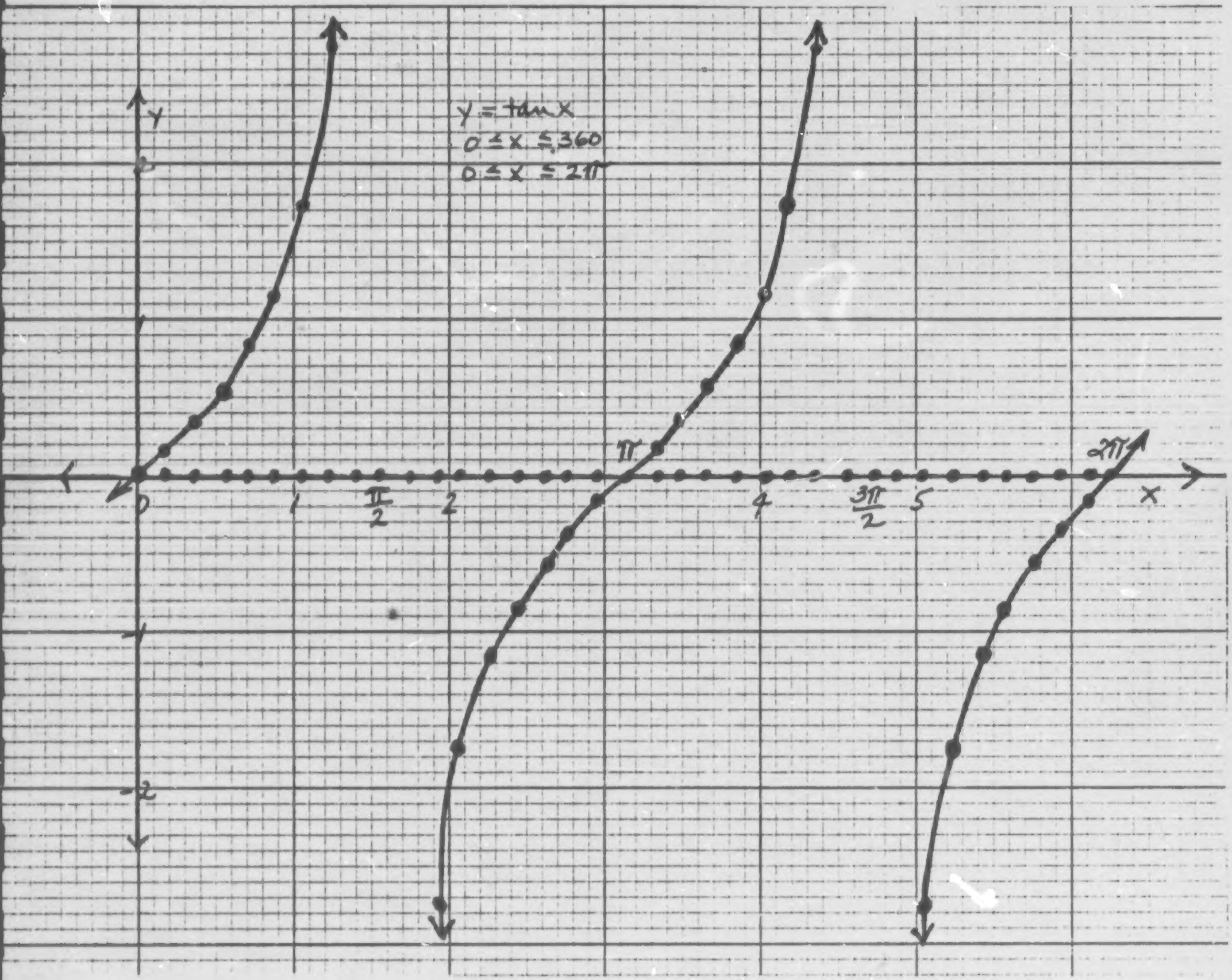
x degrees	60	70	80	90	100	110	120
x radians	$\frac{\pi}{3}$ (1.05)	$\frac{7\pi}{18}$ (1.22)	$\frac{4\pi}{9}$ (1.40)	$\frac{\pi}{2}$ (1.57)	$\frac{5\pi}{9}$ (1.75)	$\frac{11\pi}{18}$ (1.92)	$\frac{2\pi}{3}$ (2.09)
tan x	1.73	2.75	5.67	-	-5.67	-2.75	-1.73

x degrees	130	140	150	160	170	180
x radians	$\frac{13\pi}{18}$ (2.27)	$\frac{7\pi}{9}$ (2.44)	$\frac{5\pi}{6}$ (2.62)	$\frac{8\pi}{9}$ (2.79)	$\frac{17\pi}{18}$ (2.97)	π (3.14)
tan x	-1.19	-0.84	-0.58	-0.36	-0.18	0.00

x degrees	190	200	210	220	230	240
x radians	$\frac{19\pi}{18}$ (3.32)	$\frac{10\pi}{9}$ (3.49)	$\frac{7\pi}{6}$ (3.67)	$\frac{11\pi}{9}$ (3.84)	$\frac{23\pi}{18}$ (4.01)	$\frac{4\pi}{3}$ (4.19)
tan x	0.18	0.36	0.58	0.84	1.19	1.73

x degrees	250	260	270	280	290	300
x radians	$\frac{25\pi}{18}$ (4.36)	$\frac{13\pi}{9}$ (4.54)	$\frac{3\pi}{2}$ (4.71)	$\frac{14\pi}{9}$ (4.89)	$\frac{29\pi}{18}$ (5.06)	$\frac{5\pi}{3}$ (5.24)
tan x	2.75	5.67	-	-5.67	-2.75	-1.73

310 $\frac{31\pi}{18}$ (5.41)	320 $\frac{16\pi}{9}$ (5.59)	330 $\frac{11\pi}{6}$ (5.67)	340 $\frac{17\pi}{9}$ (5.93)	350 $\frac{35\pi}{18}$ (6.11)	360 2π (6.28)
-1.19	-0.84	-0.58	-0.36	-0.18	0.00



The programs from example 1 section 5.1 can again be easily modified to be appropriate for this problem.

You probably can guess what the sketch of $y = \tan x$ will look like as x varies from -2π to 2π.

Example 2: Sketch the graph of $y = \tan x$ for values of x between -2π and 2π at intervals of $\frac{\pi}{6}$.

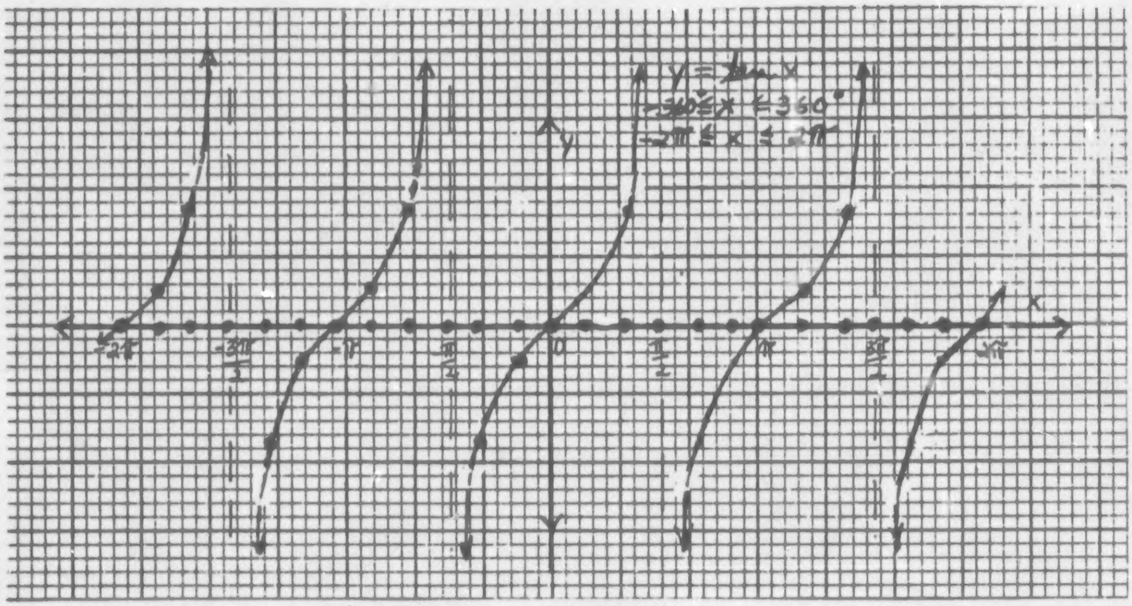
x (degrees)	-360	-330	-300	-270
x (radians)	-2π (6.28)	$\frac{11\pi}{6}$ (-5.76)	$\frac{-5\pi}{3}$ (-5.24)	$\frac{-3\pi}{2}$ (-4.71)
tan x	0.00	0.58	1.73	-

-240	-210	-180	-150	-120
$\frac{-4\pi}{3}$ (4.19)	$\frac{-7\pi}{6}$ (-3.67)	$-\pi$ (-3.14)	$\frac{-5\pi}{6}$ (-2.62)	$\frac{-2\pi}{3}$ (-2.09)
-1.73	-0.58	0.00	0.58	1.73

-90	-60	-30	0	30	60	90
$\frac{-\pi}{2}$ (-1.57)	$\frac{-\pi}{3}$ (-1.05)	$\frac{-\pi}{6}$ (-.52)	0	$\frac{\pi}{6}$ (.52)	$\frac{\pi}{3}$ (1.05)	$\frac{\pi}{2}$ (1.57)
-	-1.73	-0.58	0.0	0.58	1.73	-

120	150	180	210	240	270
$\frac{2\pi}{3}$ (2.09)	$\frac{5\pi}{6}$ (2.62)	π (3.14)	$\frac{7\pi}{6}$ (3.67)	$\frac{4\pi}{3}$ (4.19)	$\frac{3\pi}{2}$ (4.71)
-1.73	-0.58	0.00	0.58	1.73	-

300	330	360
$\frac{5\pi}{3}$ (5.24)	$\frac{11\pi}{6}$ (5.76)	2π (6.28)
-1.73	-0.58	0.00



The tangent function's characteristic shape is dramatic because it is discontinuous when $x = \frac{k\pi}{2}$ (where k is an odd integer). Because the tangent curve approaches lines such as $x = \frac{\pi}{2}$ and $x = \frac{-\pi}{2}$, but never crosses them, these lines are called asymptotes and the curve is said to be asymptotic to these lines. Because the value of the tangent function can become larger than any fixed positive number and smaller than any fixed negative number we do not define the amplitude of the tangent function.

Notice that y = tan x:

1. has a characteristic shape
2. is discontinuous whenever $x = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{3\pi}{2}, \frac{-5\pi}{2}, \frac{5\pi}{2}$, ... that is odd multiples of $\frac{\pi}{2}$
3. has all real numbers except odd multipliers of $\frac{\pi}{2}$ as its domain
4. has all real numbers as its range

5. increases in all quadrants
6. is positive in quadrants I and III
7. is negative in quadrants II and IV
8. has a period of 180° or π

Exercises 5.2

(1 - 4) From the graph, estimate the values of x, $-2\pi \leq x \leq 2\pi$ for which:

(1) $\tan x = \tan (-50)^\circ$

(2) $\tan x = .5$

(3) $\tan x = -2$

(4) $\tan x = x$

(5) Explain how you would convince another student that $\tan (180 + x) = \tan x$ from the graph of $y = \tan x$.

(6) Describe a function, other than tangent, that has an asymptote. Sketch the function and label its asymptote(s).

(7 - 8) (a) Without calculating specific values, sketch what you think the graph of each of these functions will look like.

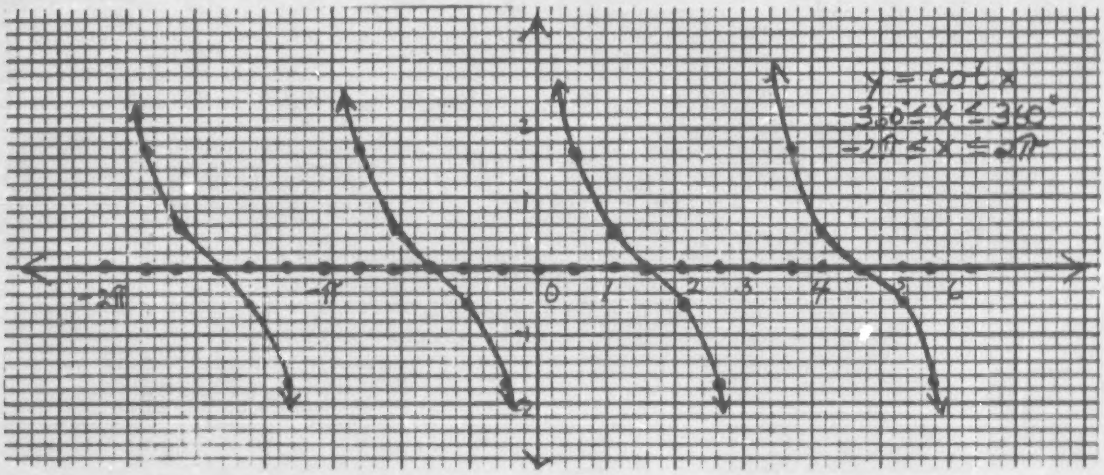
(b) Sketch each graph for values of x between 0 and 2π.

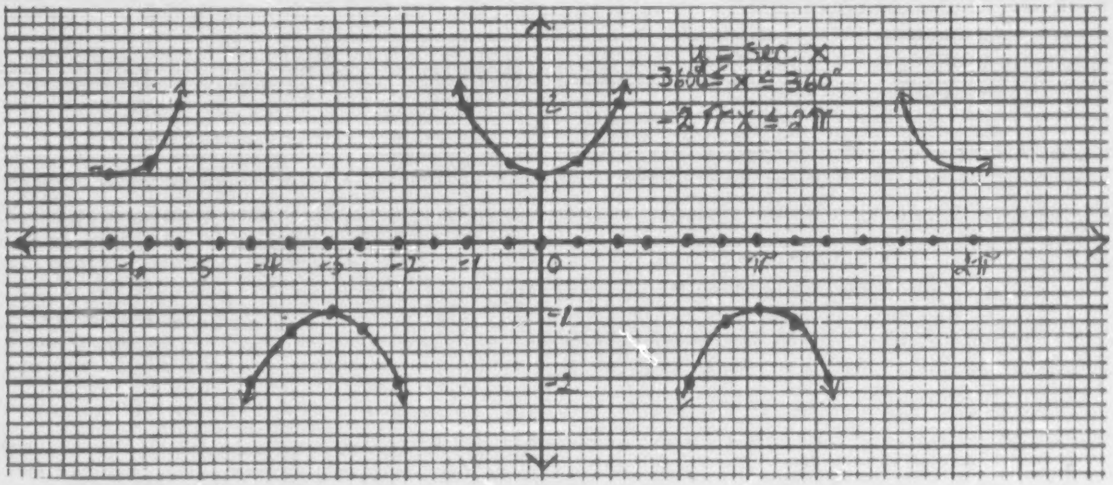
(7) $y = \sin x + \tan x$

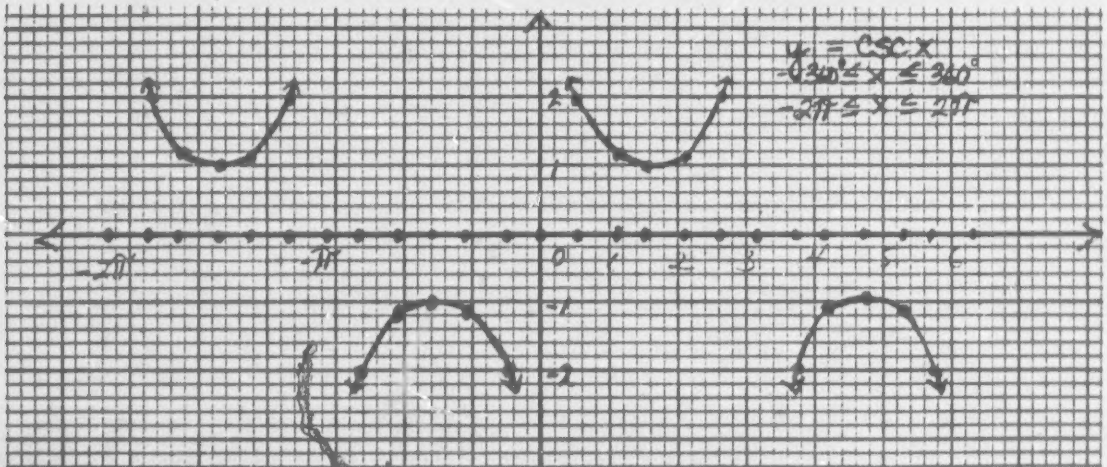
(8) $y = \tan x - \cos x$

5.3 Graphs of the Cotangent, Secant and Cosecant

The graphs of the cotangent, secant and cosecant can be drawn in the same manner as those of the sine, cosine and tangent. Sketches of these graphs are shown below.







Exercises 5.3

(1 - 3) Make a table of values and a careful sketch of each of the following for $-2\pi \le x \le 2\pi$ at intervals of $\frac{\pi}{6}$.

(1) $y = \cot x$ (2) $y = \sec x$

(3) $y = \csc x$

(4 - 9) Determine each of the following for each of the functions cot, sec, csc.

(4) period (5) domain

(6) range (7) (dis)continuity

(8) variation (9) amplitude

(10) Explain why the graphs of $y = \sec x$ and $y = \csc x$ never cross the x-axis, while graphs of the other functions do cross the x-axis.

(11 - 13) Graph each of the following pairs of functions on the same axes.

(11) $y = \sin x$, $y = \csc x$

(12) $y = \cos x$, $y = \sec x$

(13) $y = \tan x$, $y = \cot x$

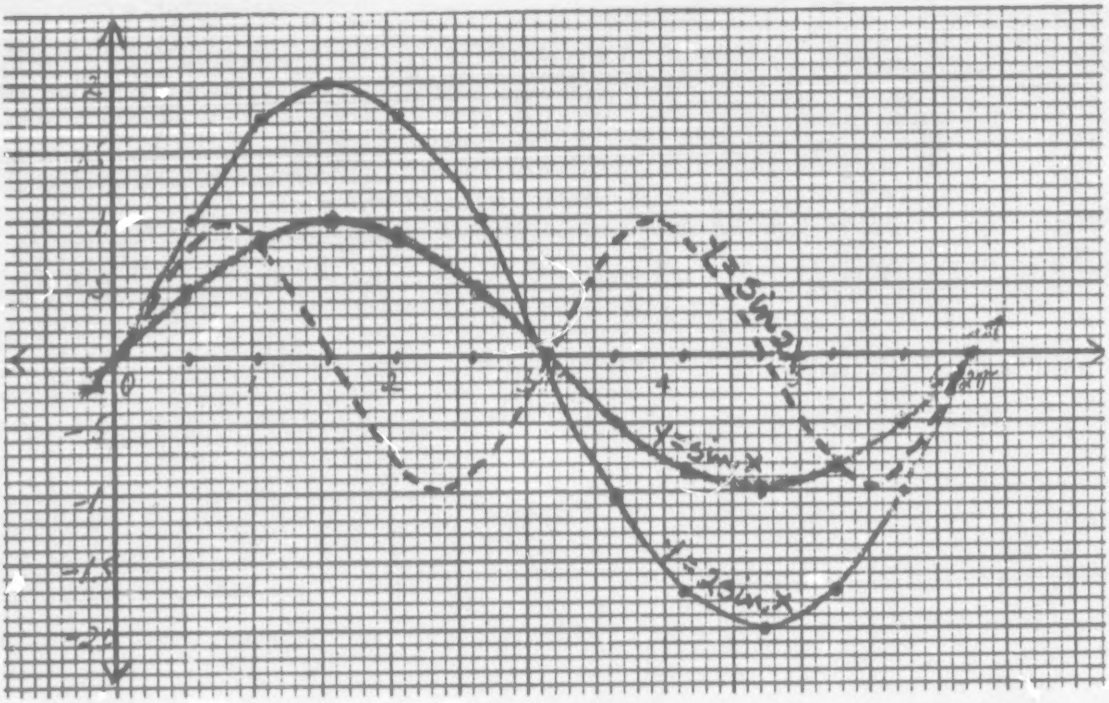
(14) Discuss how the graphs of each of (11), (12), and (13) are related.

5.4 Amplitude, Frequency and Period of the Sine and Cosine Functions

The following graph shows a sketch of three different sine curves on the same axes: $y = 2 \sin x$, $y = \sin x$, $y = \sin 2x$.

Fill in the missing entries in this table*

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
2 sin x		1.00		2.00			0.00						
sin x		0.50					0.00						
sin 2x		0.87			-0.87		0.00						



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* You may wish to find additional values, for example, $\frac{\pi}{4}$, $\frac{3\pi}{4}$, ... $\frac{7\pi}{4}$.

Notice that multiplying the value of the angle by 2 divides the period by 2 and does not change the amplitude. Notice also that multiplying the value of the function by 2 multiplies the amplitude by 2 and does not change the period.

Notice that the frequency or the number of cycles within 2π is 2 for $y = \sin 2x$, and 1 for $y = \sin x$ and $y = 2 \sin x$.

Let $f(x)$ be a sine or cosine function.

$y = a \cdot f(p \cdot x)$ has a period $\left|\frac{1}{p}\right|$ times the period of $y = f(x)$.

$y = a \cdot f(p \cdot x)$ has an amplitude $|a|$ times the amplitude of $y = f(x)$.

$y = a \cdot f(p \cdot x)$ has a frequency of $|p|$ cycles for every 2π (360°).

You should become so familiar with the basic curves $y = \sin x$, $y = \cos x$ and $y = \tan x$ that you can sketch their characteristic shape without calculating particular values. Knowing the period and amplitude of a function you should be able to quickly sketch the graph using only the values of x at 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$ and 2π.

Exercises 5.4

(1 - 10) For the given function state the a) amplitude b) period and c) frequency in both degrees and radians.

(1) $y = \sin x$

(2) $y = 2 \cos 3x$

(3) $y = \frac{1}{2} \sin 3x$　　(4) $y = 4 \sin x$

(5) $y = \frac{1}{3} \sin 4x$　　(6) $y = .7 \cos .2x$

(7) $y = \cos 5x$　　(8) $y = 5 \sin \frac{1}{2} x$

(9)[*] $y = \frac{1}{6} \tan 3x$　　(10)[*] $y = .27 \csc 6x$

(11 - 14) Write an equation for a sine curve whose period is π and whose amplitude is:

(11) 2　　(12) $\frac{1}{3}$

(13) .53　　(14) 6.2

(15 - 18) Write an equation for a cosine curve whose amplitude is 4 and whose period is:

(15) 360°　　(16) $\frac{\pi}{2}$

(17) 3π　　(18) 72°

(19 - 22) Determine the maximum and the minimum value of each of the following functions:

(19) $y = 2 \sin 3x$　　(20) $y = 2 \cos 7x$

(21) $y = \frac{1}{2} \cos 5x$　　(22) $y = 12 \sin x$

(23) Complete the following chart

function	period	amplitude	frequency
$y = \sin x$			
$y = 5 \sin x$			
$y = \sin 5x$			
$y = 5 \sin 5x$			
$y = \cos x$			
$y = 5 \cos x$			
$y = 5 \cos 5x$			

---

[*] Extend the ideas of period, amplitude and frequency to accommodate these functions.

(24) Explain why the period of $y = \sin 2x$ is $\frac{1}{2}$ the period of $y = \sin x$.

(25) How are the period and frequency of a function related?

(26) Explain why the 3 in the function $y = \sin 3x$ changes the frequency and period but not the amplitude.

(27) (a) On the same axes sketch the graph of $y = \frac{1}{2} \cos x$ and $y = \cos \frac{1}{2} x$, as x varies from 0 to 2π.

(b) Use your graph to determine the number of values that satisfy the equation $\frac{1}{2} \cos x = \cos \frac{1}{2} x$ for x between 0 and 2π inclusive.

(28) (a) On one set of axes, sketch the graphs of $y = 3 \sin x$ and $y = 3 \cos x$ from 0 to π.

(b) From the graphs made in answer to (a), determine the number of values of x between 0 and π inclusive for which $3 \sin x = 3 \cos x$.

(29) (a) Sketch the graphs of $y = \frac{1}{2} \sin x$, $0 \leq x \leq 2\pi$, and $y = .3$ on the same set of axes.

(b) Use your sketches to approximate the solution set for the equation $\frac{1}{2} \sin x = .3$.

(30) (a) On the same set of axes sketch the graphs of $y = 2 \cos x$ and $y = \sin \frac{1}{2} x$ as x varies from 0 to 2π.

(b) Determine the values of $0 \leq x \leq 2\pi$ for which $2 \cos x = \sin \frac{1}{2} x$.

(31 - 34)[*] Sketch the graph of the following trigonometric functions for all real numbers between -2π and 2π.

(31) $y = \sin \frac{1}{x}$

(32) $y = x \sin \frac{1}{x}$

(33) $y = \frac{1}{x} \sin x$

(34) $y = e^{-x} \sin x$

---

[*] These problems are unusual and challenging to even the best of students.

(35) Determine the period, amplitude, frequency and cycle of the functions in exercises (31) - (34).

(36) In some cases in exercises (31) - (34) the concepts of period, amplitude, frequency and cycle do not apply. Discuss why this is true and how you could anticipate these irregularities by carefully examining the equations of these functions.

(37) Sketch the graph of $y = 2 \sin x \cos x$ for $-2\pi \leq x \leq 2\pi$.

(38) Write another equation for the function described in (37) and verify that these two different equations represent the same function.

(39 - 41) Sketch the graphs of the following trigonometric functions for all real numbers between -2π and 2π.

(39) $y = \cos^2 \theta - \sin^2 \theta$ (40) $y = 2 \cos^2 \theta - 1$

(41) $y = 1 - 2 \sin^2 \theta$

(42) What special properties are true about the functions described in (39) - (41)? Verify your answer.

5.5 Graphs of the Inverse Trigonometric Functions

In graphing the trigonometric functions the first element of each ordered pair is the measure of an angle x expressed as a real number (radians). The second element, for example cos x, is another real number. The number x uniquely determines cos x. To obtain the inverse of the cosine function we interchange x and cos x to obtain a new set of ordered pairs $\{(\cos x, x)\}$. However, as we have already seen* cos x does not determine x uniquely. In fact there are infinitely many values of x corresponding to any specific cos x. Therefore the inverse of the cosine function is a relation** and not a function. The same is true of each of the other trigonometric functions.

If a trigonometric function is made one-to-one by restricting its domain, then its inverse will also be a function. If the domain of the cosine function is restricted to $0 \leq x \leq \pi$, there is a unique value for x. In a similar manner, the domain of the other trigonometric functions may be restricted to produce inverse functions. The restricted domains are the principal values of these functions.

To distinguish the inverse functions we have just defined from the inverse relations the initial letter of the name is capitalized*** when referring to the inverse function.

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* You may wish to look again at sections 3.4 and 4.4 and compare the discussions of inverses in these sections to the ideas developed here.

** A relation is a set of ordered pairs in which the second element is not uniquely determined by the first element. A function is a set of ordered pairs in which the second element is uniquely determined by the first element.

*** Many people indicate these differences orally by reading Arc sin x as "cap arc sine x" and $\text{Sin}^{-1} x$ as "cap sine of x inverse".

The range of the inverse trigonometric functions is:

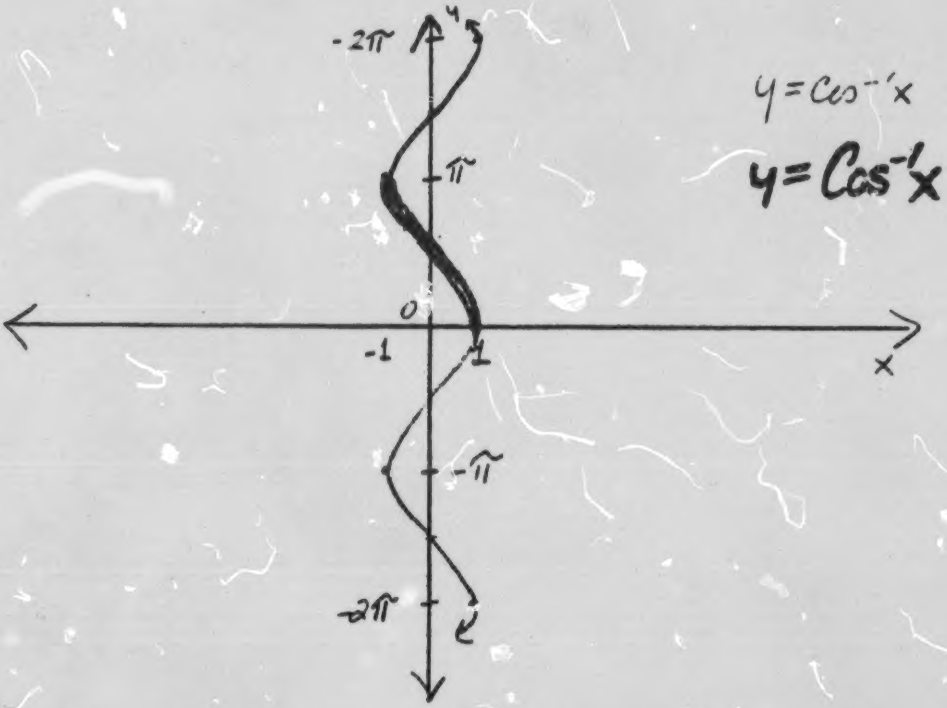
$\frac{-\pi}{2} \le \text{Arc sin } x \le \frac{\pi}{2}$ or $\frac{-\pi}{2} \le \text{Sin}^{-1} x \le \frac{\pi}{2}$

$0 \le \text{Arc cos } x \le \pi$ or $0 \le \text{Cos}^{-1} x \le \pi$

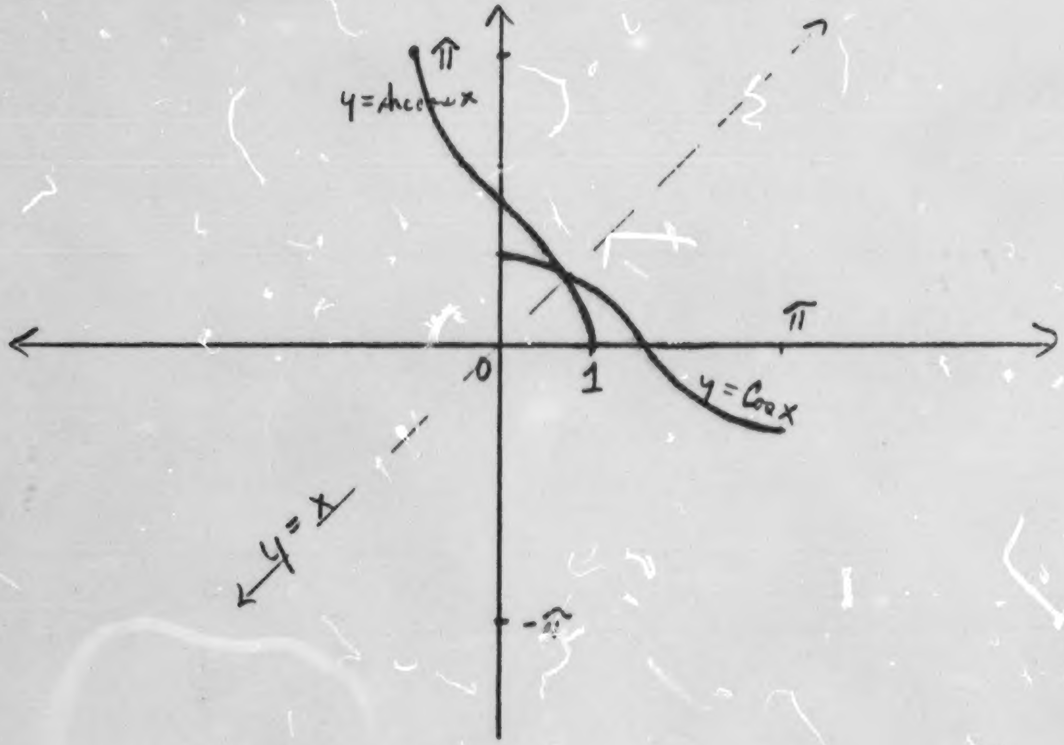
$\frac{-\pi}{2} < \text{Arc tan } x < \frac{\pi}{2}$ or $\frac{-\pi}{2} < \text{Tan}^{-1} x < \frac{\pi}{2}$

By restricing the domains of the trigonometric functions so that their inverses are also functions we can describe the function y = Sin x, y = Cos x, and y = Tan x.

Example: Sketch the graph of the relation $y = \cos^{-1} x$ and indicate the graph of the function $y = \text{Cos}^{-1} x$.



Example: Sketch the graph of $y = \text{Cos } x$ and $y = \text{Arc cos } x$ on the same axes.



Note the following summary.

Function and Inverse	Domain	Range
f: $y = \text{Sin } x$	$\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$	$-1 \leq y \leq 1$
f^{-1}: $y = \text{Sin}^{-1} x$	$-1 \leq x \leq 1$	$\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
f: $y = \text{Cos } x$	$0 \leq x \leq \pi$	$-1 \leq y \leq 1$
f^{-1}: $y = \text{Cos}^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$

Function and Inverse	Domain	Range
$f : y = \text{Tan } x$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$	$-\infty < y < +\infty$ *
$f^{-1} : y = \text{Tan}^{-1} x$	$-\infty < x < +\infty$	$\frac{-\pi}{2} < y < \frac{\pi}{2}$

Exercises 5.5

(1 - 5) Sketch the graph of each of the following relations:

(1) $y = \sin^{-1} x$

(2) $y = \tan^{-1} x$

(3) $y = \cot^{-1} x$

(4) $y = \sec^{-1} x$

(5) $y = \csc^{-1} x$

(6 - 10) Use your sketches of (1) - (5) and indicate by a thicker curve each of the following:

(6) $y = \text{Sin}^{-1} x$

(7) $y = \text{Tan}^{-1} x$

(8) $y = \text{Arc cot } x$

(9) $y = \text{Arc sec } x$

(10) $y = \text{Arc csc } x$

(11) Complete the following tables

Function and Inverse	Domain	Range
$f : y = \text{Cot } x$ $f^{-1} : y = \text{Cot}^{-1} x$		
$f : y = \text{Sec } x$ $f^{-1} : y = \text{Sec}^{-1} x$		
$f : y = \text{Csc } x$ $f^{-1} : y = \text{Csc}^{-1} x$		

---

* $-\infty$ and $+\infty$ represent negative infinity and positive infinity respectively. They are not real numbers but rather useful extensions of the real number system. $-\infty$ is less than any real number and every real number is less than $+\infty$. Thus $-\infty < x < +\infty$ for any real number x.

(12 - 16) Sketch the graphs of each of the following collection of functions on the same axes.

(12) $y = \text{Sin } x$, $y = \text{Arc sin } x$, $y = x$

(13) $y = \text{Tan } x$, $y = \text{Arc tan } x$, $y = x$

(14) $y = \text{Cot } x$, $y = \text{Cot}^{-1} x$, $y = x$

(15) $y = \text{Sec } x$, $y = \text{Sec}^{-1} x$, $y = x$

(16) $y = \text{Csc } x$, $y = \text{Csc}^{-1} x$, $y = x$

(17) Choose any function, $f(x)$, whose inverse, $f^{-1}(x)$, is also a function. Sketch the graphs of $y = f(x)$, $y = f^{-1}(x)$ and $y = x$ on the same axes.

(18) What relationship exists between the graph of a function, the graph of the inverse of the function and the graph of the line $y = x$. Why?

(19) Suppose that your calculator did not have $\boxed{\tan}$ or $\boxed{\tan^{-1}}$ keys. How could you determine values for these functions?

(20 - 25) (a) Express each of the following algebraic expressions as a keystroke sequence for your calculator. (b) Execute the sequence of keystrokes on your calculator and give a reason why your calculator displays the answer that is given.

(20) $\text{Tan}^{-1}\ 45678$ (21) $\text{Sin}^{-1}\ 45678$

(22) $\text{Cos}^{-1}\ 45678$ (23) $\text{Cot}^{-1}\ 45678$

(24) $\text{Csc}^{-1}\ 0$ (25) $\text{Sec}^{-1}\ \frac{\pi}{2}$

(26) On page 5.5 - 1 we said, "If a trigonometric function is made one-to-one by restricting the domain, then its inverse will also be a function". Why is this true?

(27) For $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$, the function $y = \sin x$ has an inverse that is also a function. Name another restricted domain for which this function has an inverse function. Give a reasonable explanation why this is not the range of $y = \text{Sin } x$.

(28) Why would it be reasonable to read Arc $\sin^{-1} x$ as "cap inverse sine of x". What does this infer about the relation of sine and the relation of inverse?

(29 - 32) Sketch the graphs of each of the following relations.

(29) $y = \text{arc sin } 3x$

(30) $y = 2 \text{ arc cos } 2x$

(31) $y = \text{arc cos } \frac{1}{2} x$

(32) $y = \frac{1}{2} \text{ arc sin } \frac{1}{2} x$

(33) Choose a function, $f(x)$, whose inverse, $f^{-1}(x)$, is not a function. Sketch the graph of $f^{-1}(x)$. Consider the family of lines $x = k$ where k is any real number. What is the value of k such that the line $x = k$ intersects the graph of $y = f^{-1}(x)$ in more than one place? Could this happen if $f^{-1}(x)$ was a function? Why?

(34 - 40) Find the value of each of the following

(a) without a calculator and
(b) verify that your results are correct by checking your answer to (a) with your calculator.

(34) tan (Arc sin 1/2)

35) sin (Arc tan 5/12)

(36) csc (Arc sin 1/2)

37) cot (Arc sin ($-\frac{\sqrt{3}}{2}$))

(38) cos (arc sin ($\frac{-\sqrt{3}}{2}$ -))

39) cos (arc csc 25/7)

(40) cot (Arc cos (-1/2)).

Chapter 5 TEST

Directions: For (1 - 3), a) Evaluate the expression to four decimal places using your calculator. b) Find the exact value of the expression without your calculator.

1) $\tan[\text{Arc}\sin(\sqrt{3}/2)]$

2) $\cos[\text{Arc}\cot(-5/12)]$

3) Given: $\cos t = -8/17$
Find: $\sec t$

(4 - 7) Find the value of each of the following expressions.

4) $(\cos 2\pi)(\sin^2 \pi/6) - (\cos \pi)(\sin \pi/3)^2$

5) $\sec \pi + \cos \pi/2 - \tan \pi/2$

6) $\cos[\text{Arc}\sin 1/2 + \text{Arc}\cos 1/2)$

7) $\text{Arc}\cos 1/2 + \text{Arc}\cos \sqrt{3}/2$

8) For what replacements of θ, such that $0 \leq \theta \leq \pi$ is $\tan \theta = \cot \theta$?

9) As t increases from $\pi/2 + \pi$, find the variation of the $\csc t$.

10) Given: $y = b \sin bx$ and $y = b \cos bx$.

For $b = 5$, how many more x-intercepts between $[0, 2\pi]$ are found for the sine function in comparison to the cosine function?

(11 - 14) Answer the following, using the letter before the function listed below. If there is more than one correct answer, list them all.

A) $y = 2 \sin 1/2x$
B) $y = 3 \cos 2x$
C) $y = 1/2 \tan 2x$
D) $y = -1/2 \sin 3x$

11) Function(s) with amplitude of $1/2$

12) Function(s) with period of $\pi/2$

13) Function(s) with frequency of 3

14) Discontinuous function(s)

15 a) Sketch the graphs of A and B from (11 - 14). Use the same set of axes in the interval $[0, 2\pi]$. Label each graph with its equation.

b) For how many values in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ does $2 \sin 1/2x = 3 \cos 2x$?

16 a) Sketch the graphs of $y = 2 \cos^2 x - 1$ and $y = -\cos x$ on the same axes in the interval $[0, 2\pi]$. Label each graph with its equation.

b) For what value(s) of x in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ is $2 \cos^2 x - 1 + \cos x = 0$?

CHAPTER 6 SOLUTION OF OBLIQUE TRIANGLES AND OTHER APPLICATIONS OF TRIGONOMETRY

In chapter 3 we solved problems represented by right triangles. In this chapter we will consider problems represented by triangles that are not right triangles. Such triangles are called *oblique* triangles and the process of finding the measures of all of sides and angles of a triangle when only some of these measures are known is called *solving the triangle*. We will also consider some of the other applications of trigonometry.

6.1 Oblique Triangles

Recall from plane geometry that a triangle is uniquely determined if we know:

- two angles and one side (ASA or AAS)
- two sides and an included angle (SAS)
- three sides (SSS).

In general, knowing the measures of two sides and an opposite angle of a triangle (SSA) is not sufficient to determine a unique triangle. It sometimes happens that only one triangle, two triangles or no triangle can be constructed given the measures of two sides and an opposite angle of a triangle. Since there are three possible outcomes to this situation, (SSA), is referred to as the *ambiguous case*.

In making our analysis of these situations we shall refer to the known angle as $\measuredangle X$ and the known sides as x and y.[*]

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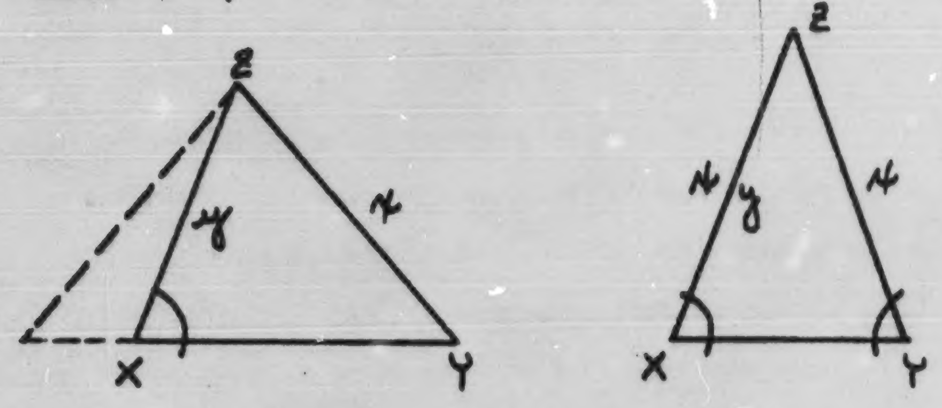
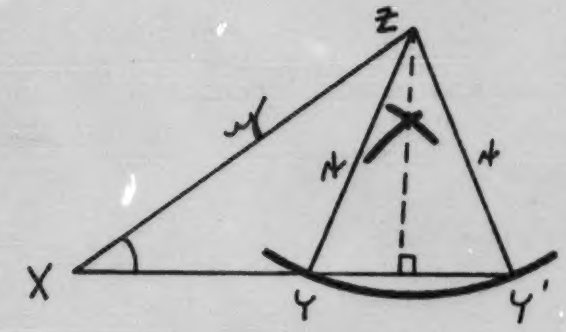
[*] We will use the customary notation in which an angle is denoted by a capital letter and the side opposite is denoted by the lower case of the same letter.

Case A $\measuredangle X$ is acute and $x \geq y$.

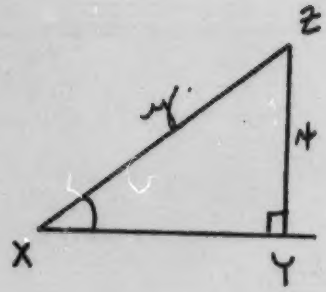


Only one triangle is possible.

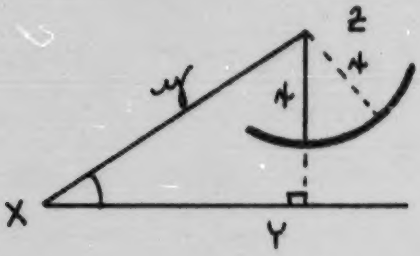
Case B $\measuredangle X$ is acute and $x < y$.



Two triangles $\triangle XZY'$ and $\triangle XZY$ are possible.



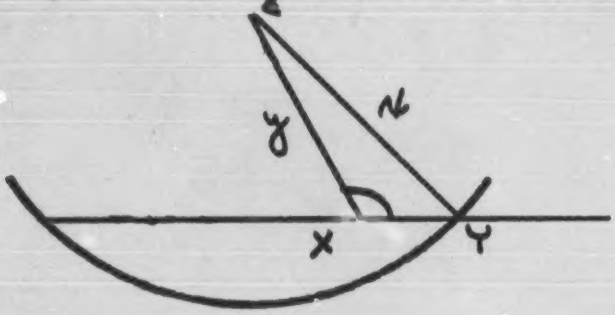
One triangle is possible and $\overline{YZ} \perp \overline{XY}$.



No triangle is possible since x is shorter than the altitude from $\overline{XZ}$ to $\overline{XY}$.

Case C

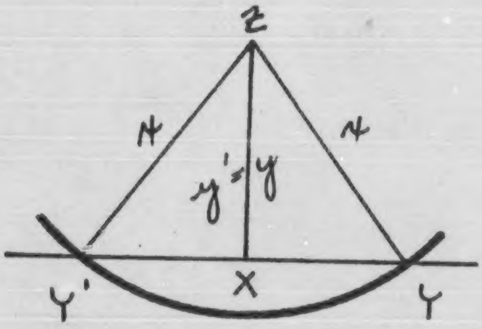
$\measuredangle X$ is obtuse and $x > y$.



Only one triangle is possible.

If $\measuredangle X$ is obtuse, then $m\measuredangle X > m\measuredangle Y$ so it is impossible that $x \leq y$.

Case D



$\triangle XY'Z \cong \triangle XYZ$

If $\measuredangle X$ is right, then $m\measuredangle X > m\measuredangle Y$ and $x > y$ so one triangle is possible. If $\measuredangle X$ is right then $m\measuredangle X > m\measuredangle Y$ so it is impossible that $x \leq y$.

Summary of the ambiguous case

$\measuredangle X$ is acute and

$x \geq y$	one triangle
$x < y$	two triangles ($x >$ altitude to y) one (right) triangle ($x =$ altitude to y) no triangle ($x <$ altitude to y)

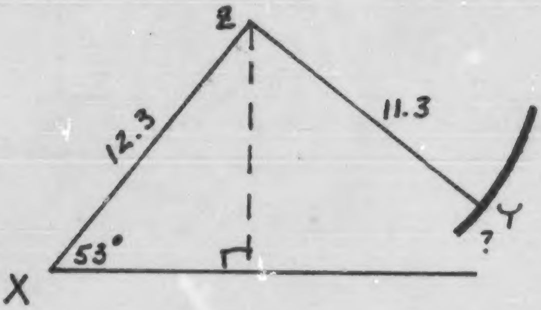
$\measuredangle X$ is obtuse or right

$x > y$	one triangle
$x \leq y$	no triangle

Notice that whenever the given angle is opposite the larger of the two given sides only one triangle is possible.

Example

Determine the number of possible solutions when $m \angle X = 53^\circ$ and $x = 11.3$, $y = 12.3$.



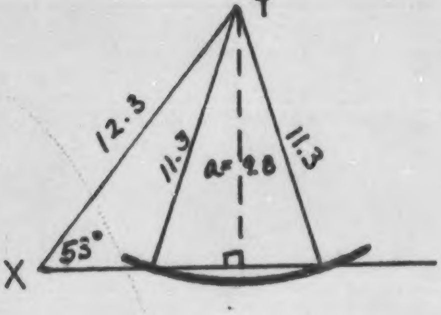
We need to know the length of the altitude from Y to $\overline{XZ}$.

Consider $\triangle$ XYZ' where Z' is a right angle.

Let a = altitude from Y to $\overline{XZ}$.

$$\sin 53^\circ = \frac{a}{12.3}$$

$$9.8 \approx a$$



since $11.3 > 9.8$ there are two possible triangles.

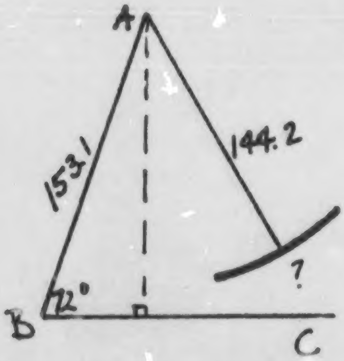
Example

How many triangles ABC can be constructed so that $m \angle B = 72^\circ$, AB = 153.1 and AC = 144.2?

let h = altitude from A to $\overline{BC}$

$$\sin 72 = \frac{h}{153.1}$$

$$145.6 = h$$



since $h > AC$ no such triangle can be constructed.

Exercises 6.1

(1) After case C we said "If $\measuredangle X$ is obtuse then $m\measuredangle X > m\measuredangle Y$ so it is impossible that $x \leq y$". Why?

(2) Why is it unnecessary to consider a case when $\measuredangle X$ is a right angle?

(3 - 8) Find the number of possible triangles XYZ that could be constructed if:

(3) $m\measuredangle X = 14^\circ$, $YZ = 7.3$, $XY = 5.2$

(4) $m\measuredangle Y = 103^\circ\ 14'$, $XZ = 93.1$, $YZ = 98.2$

(5) $m\measuredangle Z = 52.68^\circ$, $XY = 93.1$, $YZ = 98.2$

(6) $m\measuredangle X = 87^\circ$, $XY = 9$, $YZ = 8$

(7) $m\measuredangle X = 62.73^\circ$, $XY = 9$, $YZ = 8$

(8) $m\measuredangle X = 7^\circ$, $XY = 9$, $YZ = 8$

(9) Look carefully at exercises 6 - 8 above and make a statement about the relationship between the lengths of sides and the measure of the angle opposite one of the sides.

6.2 The Law of Sines

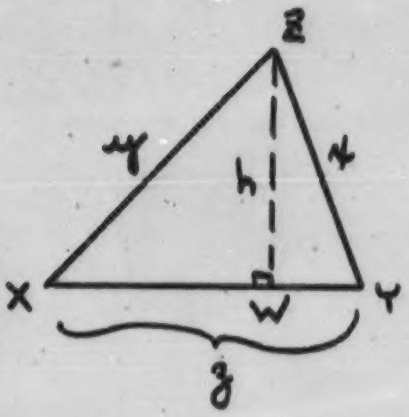
In an oblique triangle if two angles and one side (ASA or AAS) or two sides and the angle opposite one of them (SSA) are known, then the solution of the triangle may be determined by using the LAW OF SINES which states that: the measures of the sides of a triangle are proportional to the sines of the measures of angles opposite the sides.

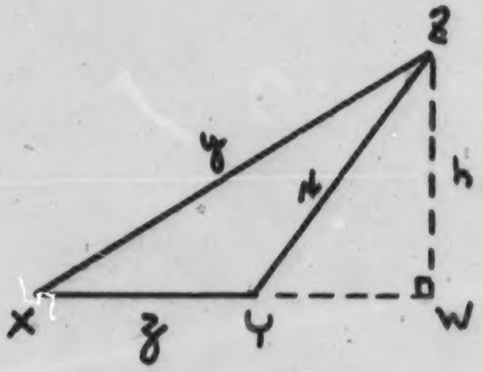
LAW OF SINES: For each oblique triangle XYZ,

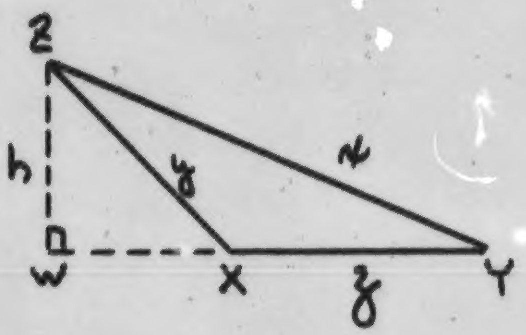
$$\frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z} .$$

Proof: Given oblique $\triangle XYZ$, let h be the altitude from Z to $\overline{XY}$.

$\measuredangle X$ is either acute or obtuse as shown below.







In each case for right $\triangle YZW$

$$\sin Y = \frac{h}{x} \quad \text{so} \quad h = x \sin Y$$

and for right $\triangle XWZ$

$$\sin X = \frac{h}{y} \quad \text{so} \quad h = y \sin X$$

and $x \sin Y = y \sin X$

thus $\frac{x}{\sin X} = \frac{y}{\sin Y}$

If we let h be the altitude from Y we would obtain

$$\frac{x}{\sin X} = \frac{z}{\sin Z}$$

thus: $\frac{x}{\sin X} = \frac{y}{\sin Y} = \frac{z}{\sin Z}$

Example 1 Solve $\triangle XYZ$ if $m\angle X = 48.3^\circ$, $m\angle Y = 27.1^\circ$ and y 11.3

$$\frac{x}{\sin X} = \frac{y}{\sin Y}$$

$$\frac{x}{\sin 48.3^\circ} = \frac{11.3}{\sin 27.1^\circ}$$

(1) $x = \frac{11.3(\sin 48.3^\circ)}{\sin 27.1^\circ}$

$x \doteq 18.5$ cm.

$m\angle Z = 104.6^\circ$

$$\frac{y}{\sin Y} = \frac{z}{\sin Z}$$

$$\frac{11.3}{\sin 27.1} = \frac{z}{\sin 104.6^\circ}$$

(2) $$\frac{11.3\ (\sin 104.6)}{\sin 27.1} = z$$

$$24.0 \doteq z$$

Notice the similarity between the equation (1) for x and the equation (2) for z. If you have a programmable calculator you may wish to write a program that can be used for solving oblique triangles using the law of sines.

Example 2 Solve $\triangle ABC$ given $m \measuredangle A = 42.5^\circ$, $a = 14.7$, and $b = 24.6$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

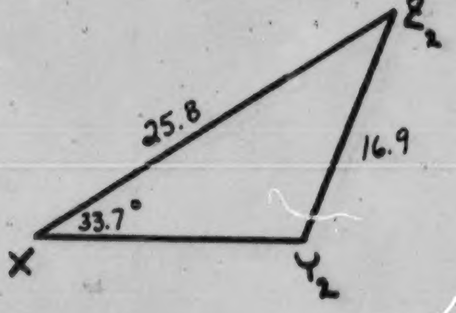
$$\sin^{-1}\left(\frac{b \sin \angle A}{a}\right) = m \measuredangle B$$

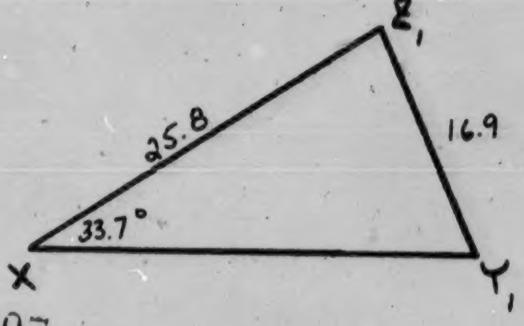
$$\sin^{-1}\left(\frac{24.6 \sin 42.5^\circ}{14.7}\right) = m \measuredangle B$$

$$\sin^{-1}(1.13) = m \measuredangle B$$

Since $\sin B > 1$ there is no triangle that satisfies the conditions of this example.

Example 3 Draw sketches of two triangles XY_1Z_1 and XY_2Z_2 for which $m \measuredangle X = 33.7^\circ$, $x = 16.9$ and $y = 25.8$, then solve both triangles.





$$\frac{16.9}{\sin 33.7} = \frac{25.8}{\sin Y_1}$$

$$\sin Y_1 = \frac{25.8 \sin 33.7}{16.9}$$

$$m \measuredangle Y_1 = 57.9$$

$\measuredangle Y_1$ is acute, $\measuredangle Y_2$ is obtuse

$$m \measuredangle Y_2 = 180 - 57.9 = 122.1^\circ$$

$$m \measuredangle Z_1 = 180^\circ - (33.7 + 57.9)^\circ = 88.4^\circ$$

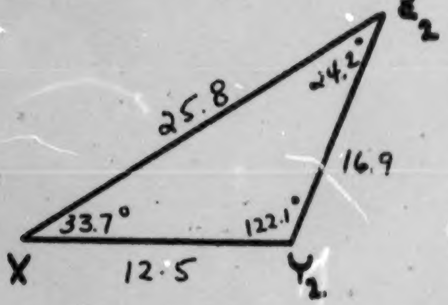
$$\frac{16.9}{\sin 33.7^\circ} = \frac{z_1}{\sin 88.4^\circ}$$

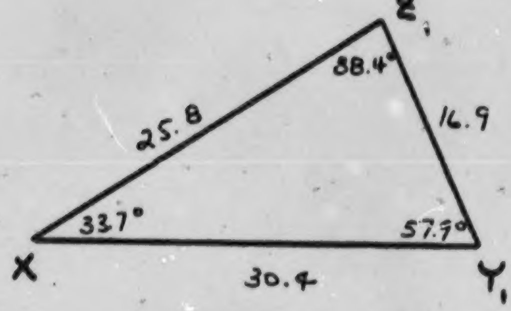
$$30.4 \doteq z_1$$

$$m \measuredangle Z_2 = 180^\circ - (33.7 + 122.1)^\circ = 24.2^\circ$$

$$\frac{16.9}{\sin 33.7} = \frac{z_2}{\sin 24.2^\circ}$$

$$12.5 \doteq z_2$$





Exercises 6.2

(1) Let h be the altitude to side y of $\triangle XYZ$. Prove that $h = z \sin X = x \sin Z$.

(2) In $\triangle XYZ$ prove that $\frac{1}{2} xy \sin Z = \frac{1}{2} xz \sin Y = \frac{1}{2} yz \sin X$. (Hint: show that the area of $\triangle XYZ = \frac{1}{2} xy \sin Z$.)

(3) Prove the Law of Sines by using the equation established in exercise 2.

(4) In $\triangle XYZ$ why is $\frac{\sin X}{x} = \frac{\sin Y}{y} = \frac{\sin Z}{z}$?

(5) Prove that the Law of Sines is true for right triangles.

(6) It is possible to express the Law of Sines by saying, "Within any triangle the ratio of the sine of an angle to the length of its opposite side is constant." Explain why this form of the Law of Sines is equivalent to the form we have used in this section.

In exercises (7 - 24) you may wish to write a calculator program that can be used to solve groups of problems.

(7 - 12) Solve triangle XYZ given the following sides and angles.

(7) $m\measuredangle X = 143.2^\circ$, $y = 11.3$, $m\measuredangle Y = 29.5^\circ$

(8) $m\measuredangle Y = 119^\circ$, $x = 112.37$, $m\measuredangle Z = 30^\circ$

(9) $m\measuredangle Y = \frac{\pi}{5}$, $y = 26$, $m\measuredangle Z = \frac{5\pi}{7}$

(10) $m\measuredangle X = 123^\circ$, $y = 7.5$, $m\measuredangle Z = 15.7^\circ$

(11) $m\measuredangle X = 72^\circ\ 15'$, $z = 23.87$, $m\measuredangle Z = 42^\circ\ 57'$

(12) $m\measuredangle X = 136^\circ\ 10'$, $y = 58.4$, $m\measuredangle Z = 46^\circ\ 25'$

(13 - 16) Use the formula of exercise 2 to determine the area of $\triangle ABC$ given the following information.

(13) $m\measuredangle A = 28^\circ$, $b = 11$, $c = 7$

(14) $m\measuredangle A = 148^\circ$, $b = 22.7$, $c = 17.5$

(15) $m\measuredangle C = 38^\circ$, $a = 12.26$, $b = 15.73$

(16) $m\measuredangle B = 109^\circ$, $a = 7.25$, $c = 8.25$

(17) In $\triangle ABC$, $a = 146.2$, $b = 87.7$ and the area of $\triangle ABC$ is 312.8. Determine $m \measuredangle C$.

(18 - 19) For each of the following sketch two triangles XY_1Z_1 and XY_2Z_2 and solve both triangles.

(18) $m \measuredangle X = 38^{\circ}$, $y = 5.8$, $x = 4.7$

(19) $m \measuredangle X = 24^{\circ}$, $y = 8.43$, $x = 5.73$

(20) The longer diagonal of a rhombus is 12 centimeters long and forms an angle of 38° with the sides. Determine the lengths of the sides of the rhombus.

(21 - 24) Solve triangle MPQ given the following information.

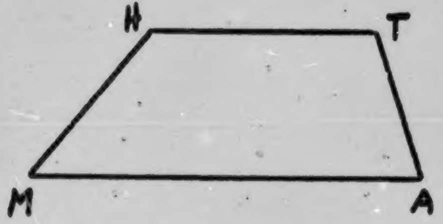
(21) $m \measuredangle M = 28^{\circ}$, $m = 19$, $p = 34$

(22) $m \measuredangle M = 125^{\circ}\ 14'$, $m = 11.56$, $p = 9.37$

(23) $m \measuredangle Q = 67^{\circ}\ 53'$, $q = 121.4$, $p = 123.1$

(24) $m \measuredangle P = 43.7^{\circ}$, $p = 18.7$, $m = 12.6$

(25) In trapezoid MATH, $m \measuredangle M = 72^{\circ}\ 18'$ $M \measuredangle A = 42^{\circ}\ 15'$, $MA = 12.18$ and $HT = 5.17$. Find the lengths of sides $\overline{MH}$ and $\overline{TA}$. (Hint: draw $\overline{TB} \parallel \overline{MH}$.)



(26) Look back at exercise (20) on page 3.5 - 6. Use a method other than your original solution to solve this problem.

6.3 The Law of Cosines

In an oblique triangle if two sides and the included angle (SAS) or three sides (SSS) are known then the solution of the triangle may be determined by using the LAW OF COSINES. This law has three forms which are essentially the same.

LAW OF COSINES

For any triangle XYZ,

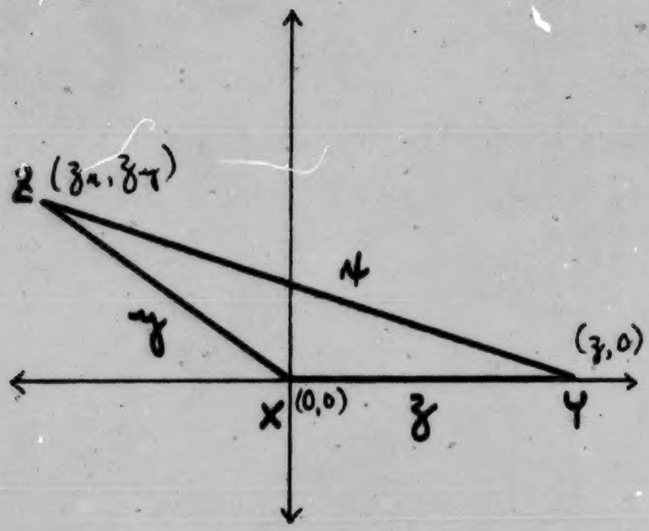
$$x^2 = y^2 + z^2 - 2yz \cos X$$

$$y^2 = x^2 + z^2 - 2xz \cos Y$$

$$z^2 = x^2 + y^2 - 2xy \cos Z$$

Proof: (Analytic)

Let triangle XYZ have ∡ X position in the coordinate plane and let Y have coordinates (z, 0). Let Z have coordinates



(z_x, z_y).

$$z_x = y \cos X; \quad z_y = y \sin X$$

Using the distance formula

$$x = \sqrt{(y \cos X - z)^2 + (y \sin X)^2}$$

$$x^2 = (y \cos X - z)^2 + (y \sin X)^2$$

$$x^2 = y^2 \cos^2 X - 2yz \cos X + z^2 + y^2 \sin^2 X$$

$$y^2 \cos^2 X + y^2 \sin^2 X = y^2(\cos^2 X + \sin^2 X) = y^2$$

$$x^2 = y^2 + z^2 - 2yz \cos X$$

by merely changing the lettering in the diagram above the other two forms of the law can be verified.

Example: Solve $\triangle XYZ$ given $x = 15$, $z = 29$ and $m\angle Y = 103^\circ$

The appropriate form of the law of cosines is

$$y^2 = x^2 + z^2 - 2\,xz \cos Y$$

$$y^2 = 15^2 + 29^2 - 2(15)(29) \cos 103$$

and $y \doteq 36$

To find $m\angle X$ and $m\angle Z$ we can use the law of sines $\dfrac{y}{\sin Y} = \dfrac{z}{\sin Z}$

$$\frac{36}{\sin 103} = \frac{29}{\sin Z}$$

$${}^{*}52^\circ \doteq m\angle Z$$

$$m\angle X = 180^\circ - (103 + 52)^\circ = 25^\circ$$

---

* You may compute $m\angle Z = 53^\circ$ if you did not round y to 36.

Example: Solve triangle MPQ where $m = 10$, $p = 15$ and $q = 17$

$$m^2 = p^2 + q^2 - 2pq \cos M$$

$$10^2 = 15^2 + 17^2 - 2(15)(17) \cos M$$

$$\frac{10^2 - 15^2 - 17^2}{-2(15)(17)} = \cos M$$

$$.8117 \doteq \cos M$$

$$36^\circ \doteq m \measuredangle M$$

$$p^2 = m^2 + q^2 - 2mq \cos P$$

$$15^2 = 10^2 \_ 17^2 - 2(10)(17) \cos P$$

$$\frac{15^2 - 10^2 - 17^2}{12(10)(17)} = \cos P$$

$$.4824 \doteq \cos P$$

$$61^\circ \doteq cm \measuredangle P$$

$$m \measuredangle Q = 180^\circ - (36 + 61)^\circ = 83^\circ$$

Exercises 6.3

(1) Let triangle XYZ have $\measuredangle Y$ in standard position in the coordinate plane. Derive the following form of the law of cosines:

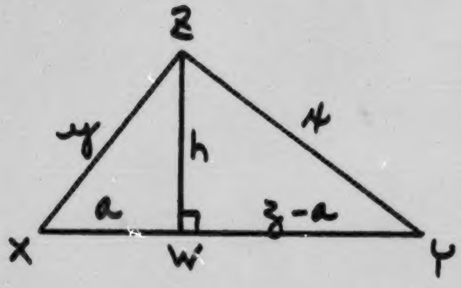
$y^2 = x^2 + z^2 - 2xz \cos Y$

(2) Determine a special form of the Law of Cosines for right triangles.

(3) Prove the converse of the Pythagorean Theorem: that is, prove that if $x^2 + y^2 = z^2$ then $\triangle XYZ$ is a right triangle having a right angle at Z.

(4) Explain one possible reason why the Pythagorean theorem is not a corollary to the Law of Cosines.

(5) Let $\overline{ZW}$ be the altitude from Z to $\overline{XY}$ in $\triangle$ XYZ. Use this information to algebraically prove the Law of Cosines for acute triangles.



(6) Prove the Law of Cosines algebraically for obtuse triangles. (Hint: let $\triangle$ XYZ have an obtuse angle at x. Let $\overline{ZW}$ be the altitude from Z to $\overline{XY}$. Consider right triangles ZWX and ZWY.

In exercises (7 - 20) you may wish to write a calculator program that can be used to solve groups of problems.

(7 - 12) Solve each $\triangle$ RST, given the following information.

(7) $m\measuredangle R = 57^\circ$, $s = 11.3$, $t = 14.7$

(8) $m\measuredangle S = 123.2^\circ$, $r = 13.37$, $t = 21.54$

(9) $m\measuredangle T = 32^\circ\ 15$, $r = 23.7$, $s = 11.5$

(10) $m\measuredangle R = 114.56^\circ$, $s = 14.00$, $t = 10.07$

(11) $m\measuredangle S = 134^\circ$, $r = 3.27$, $t = 2.14$

(12) $m\measuredangle T = 17.2^\circ$, $r = 45.7$, $s = 53.2$

(13 - 18) Determine correct to the nearest tenth the measures of each of the angles for each triangle PQR, given the following information:

(13) $p = 14.7$, $q = 16.3$, $r = 20.1$

(14) $p = 3$, $q = 3$, $r = 1.8$

(15) $p = 2.7$, $q = 4.23$, $r = 2.51$

(16) $p = 5$, $q = 14$, $r = 17$

(17) $p = 123$, $q = 90$, $r = 42$

(18) $p = 5\sqrt{2}$, $q = 6\sqrt{7}$, $r = 19.83$

(19) Determine the length of each diagonal of a parallelogram having sides of 10 and 14.7 and one angle measuring 137.2°.

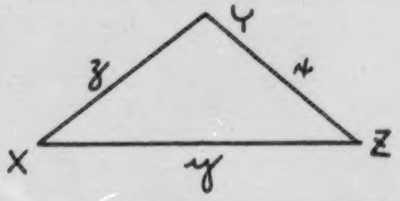
(20) The radius of a circle is 18. Find the central angle to the nearest degree subtended by a chord of length 33.

6.4 The Law of Tangents

In an oblique triangle if the measures of two sides and the included angle (SAS) are known, then the solution of the triangle may be obtained by using the LAW OF TANGENTS. This law has six forms which are essentially the same.

The LAW OF TANGENTS

$$\frac{x - y}{x + y} = \frac{\tan \frac{1}{2}(X - Y)}{\tan \frac{1}{2}(X + Y)}$$



Proof

The Law of Sines $\frac{x}{\sin x} = \frac{y}{\sin Y}$,

can be rewritten as $\frac{x}{y} = \frac{\sin X}{\sin Y}$

$$\frac{x}{y} - 1 = \frac{\sin X}{\sin Y} - 1$$

$$\frac{x}{y} - \frac{y}{y} = \frac{\sin X}{\sin Y} - \frac{\sin Y}{\sin Y} \qquad y \neq 0, \sin Y \neq 0$$

$$(1) \qquad \frac{x - y}{y} = \frac{\sin X - \sin Y}{\sin Y}$$

$$\frac{x}{y} + \frac{y}{y} = \frac{\sin X}{\sin Y} + \frac{\sin Y}{\sin Y}$$

$$(2) \qquad \frac{x + y}{y} = \frac{\sin X + \sin Y}{\sin Y}$$

combining equations (1) and (2) by division

$$\frac{x - y}{x + y} = \frac{\sin X - \sin Y}{\sin X + \sin Y}$$

now[*] $\sin X - \sin Y = 2\cos\frac{1}{2}(X+Y)\cdot\sin\frac{1}{2}(X-Y)$

$\sin X + \sin Y = 2\sin\frac{1}{2}(X+Y)\cdot\cos\frac{1}{2}(X-Y)$

thus $$\frac{x-y}{x+y} = \frac{2\cos\frac{1}{2}(X+Y)}{2\sin\frac{1}{2}(X+Y)} \cdot \frac{\sin\frac{1}{2}(X-Y)}{\cos\frac{1}{2}(X-Y)}$$

$$= \frac{2}{2}\cdot\cot\frac{1}{2}(X+Y)\cdot\tan\frac{1}{2}(X-Y)$$

$$= \frac{1}{\tan\frac{1}{2}(x+y)}\cdot\tan\frac{1}{2}(X-Y)$$

$$= \frac{\tan\frac{1}{2}(x-y)}{\tan\frac{1}{2}(X+Y)}.$$

This law has many equivalent forms that can be derived by renaming the angles of the triangle and using the properties of proportions.

Notice that the Law of Cosines easily allows us to obtain the value of the third side of a triangle when the measures of two sides and the included angle are known. Under the same conditions, the Law of Tangents gives us the measures of the two remaining angles.

Example

In $\triangle XYZ$, $x = 15$, $y = 23$ and $m\angle Z = 72^\circ$. Find $m\angle A$ and $m\angle B$. Since $y > x$ we will rewrite the Law of Tangents as

$$\frac{y-x}{y+x} = \frac{\tan\frac{1}{2}(Y-X)}{\tan\frac{1}{2}(Y+X)}$$

$$m\angle Y + m\angle X = 180^\circ - 72^\circ = 108^\circ$$

---

[*] the proofs of these statements will be developed in the exercises.

$$\frac{23 - 15}{23 + 15} = \frac{\tan \frac{1}{2} (Y - X)}{\tan \frac{1}{2} (108)}$$

$$\frac{8 \tan 54^\circ}{38} = \tan \frac{1}{2} (Y - X)$$

$$0.29 = \tan \frac{1}{2} (Y - X)$$

$$16.16^\circ = \frac{1}{2} (Y - X)$$

$$32.32^\circ = Y - X$$

$$\begin{array}{r} X + Y = 108^\circ \\ -X + Y = 32.32^\circ \\ \hline 2Y = 140.32^\circ \end{array} \qquad m \measuredangle X = 108 - 70 \doteq 38^\circ$$

$$m \measuredangle Y \doteq 70^\circ$$

Exercises 6.4

(1) Why is the Law of Tangents not appropriate when the measures of the legs and vertex angle of an isosceles triangle in known?

(2) Why is the Law of Tangents not appropriate when the measures of the legs of a right △ are known?

(3 - 8) You may wish to write a program that can be used to solve these exercises.

(3 - 8) Solve each of the following triangles XYZ if the following information is known.

(3) $x = 6.37$, $y = 8.52$, $m \measuredangle Z = 34^\circ\ 15'$

(4) $y = 14.732$, $z = 9.560$, $m \measuredangle X = 42.7^\circ$

(5) $z = 112.2$, $x = 96.2$, $m \measuredangle Y = 118^\circ$

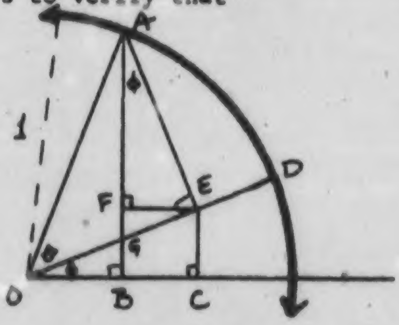
(6) $m \measuredangle X = 136.2^\circ$, $y = 15.6$, $z = \frac{37}{3}$

(7) $m \measuredangle Y = 82.7^\circ$, $x = .31$, $z = .023$

(8) $m \measuredangle Z = 9^\circ$, $x = 11.2$, $y = 9.7$

(9 - 16) Complete the following statements to verify that

$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$



$\overline{OA}$ is the radius of a unit circle and $\measuredangle AOB = \phi + \theta$

$\overline{AB} \perp \overline{OC}$, $\overline{EC} \perp \overline{OC}$, $\overline{AF} \perp \overline{FE}$,

$\overline{AE} \perp \overline{OE}$.

(9) $\triangle GOB \sim \triangle GAE$ because ______________.

(10) $\measuredangle EAG \cong \measuredangle BOG$ because ________________.

(11) $\sin\phi = \frac{EC}{\quad}$ so $EC =$, $\cos\phi = \frac{OC}{\quad}$ so $OC =$.

(12) $\sin\phi = \frac{FE}{\quad}$ so $FE =$, $\cos\phi = \frac{AF}{\quad}$ so $AF =$.

(13) $\sin\theta = \frac{AE}{1}$ so $\sin\theta =$, $\cos\theta = \frac{OE}{\quad}$ so $OE =$.

(14) $\sin(\phi + \theta) = AB = AF +$ __________ $= AF + EC$

(15) $\sin(\phi + \theta) =$ (finish by yourself).

(16 - 19) Complete the following statements to verify that

$\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta$.

(16) $\sin(\phi - \theta) = \sin\left[\phi + (\quad)\right]$

(17) $\sin(\phi - \theta) = \sin\phi \cdot$ __________ $+ \cos\phi \cdot$ _________

(18) $\cos(-\theta) = \cos$ _____; $\sin(-\theta) =$ _______

(19) $\sin(\phi - \theta) =$ (finish yourself).

(20 - 24) Complete the following statements to verify that

$\sin \phi - \sin \theta = 2 \cos \frac{1}{2} (\phi + \theta) \sin \frac{1}{2} (\phi - \theta)$ and

$\sin \phi + \sin \theta = 2 \sin \frac{1}{2} (\phi + \theta) \cos \frac{1}{2} (\phi + \theta)$. Let $A + B = \phi$ and

$A - B = \theta$.

(20) $2A =$ ____________________, $2B =$ ____________________.

(21) $A =$ ____________________, $B =$ ____________________.

(22) $\sin (A + B) + \sin (A - B) =$ _________________________.

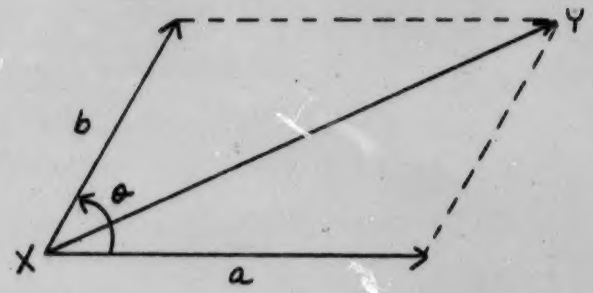
(23) $\sin (A + B) - \sin (A - B) =$ _________________________.

(24) $\sin \phi + \sin \theta =$ (finish by yourself.)

$\sin \phi - \sin \theta =$ "

6.5 Parallelogram of Forces and Other Applications of Trigonometry

When two forces are simultaneously acting on an object in two different, but not directly opposite, directions the result is the same as one force acting on the object from a direction somewhere between the original two forces. If two known forces, a and b, are acting at an angle of measure in object X, the resulting force can be determined by computing* the diagonal of a parallelogram having adjacent sides of length a and b that include an angle whose measure is θ. In our diagram the magnitude of the resulting force is the length of diagonal $\overline{XY}$. The forces a and b are known as components and force XY is called their resultant. The process of finding the resultant of two or more** forces is called determining the parallelogram of forces. The forces themselves are called vectors because they have a specific magnitude and direction.



Example: An airplane is flying at a speed of 450 kph. At the same time the wind is moving at an angle of 25.7° with the path of the plane. If the speed of the wind is 18.7 kph, how long will it take the plane to travel 719 kilometers with the wind?

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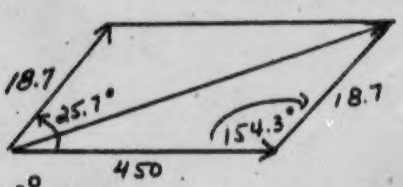
[*] It is possible to solve problems of this type by scale drawings.

[**] The resultant of three or more forces is determined by considering the forces two at a time.



Using the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$c^2 = 450^2 + 18.7^2 - 2(450)(18.7) \cos 154.3^\circ$$

$$c^2 = 218014.82$$

$$c = 466.92 \text{ kph}$$

719: x hr. = 466.92 : 1 hr.

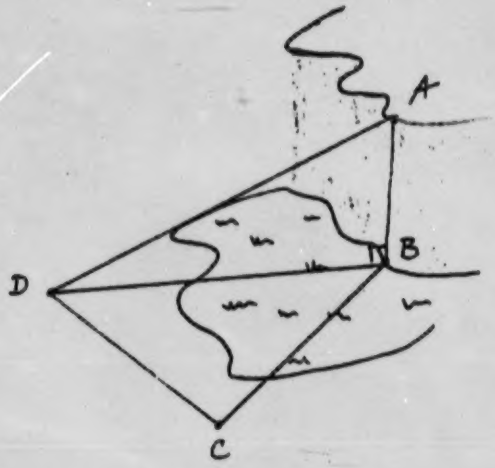
x = 1.54 hours = 1 hr. 32 min. 24 sec.

Trigonometry has many other applications especial sciences. Periodic functions like wave motion and circular functions like harmonic motion involve ideas from trigonometry that have significant application in the explanation of natural phenomena. Trigonometry is also widely used wherever indirect measurements are needed as in navigation and surveying.

We have been concerned with problems in which the angles and lines are confined to one plane. In many practical situations it is necessary to consider three dimensions.

Example:

A geology team wishes to determine the height AB of a vertical cliff whose base is inaccessible because of a swamp. They select two points C and D on solid ground and determine by instrumentation that DC = 55.3 meters,



$m\measuredangle ADB$* $= 57.23^\circ$, $m\measuredangle BDC = 23.71^\circ$ and $m\measuredangle DCB = 107.53^\circ$.

In $\triangle DBC$, $m\measuredangle DBC = 180^\circ - (83.71 + 107.53)^\circ = 48.76^\circ$

$$\frac{\sin\measuredangle DBC}{DC} = \frac{\sin\measuredangle C}{DB}$$

$$\frac{\sin 48.76^\circ}{55.3} = \frac{\sin 107.53^\circ}{DB}$$

$$70.13 \text{ m} = DB$$

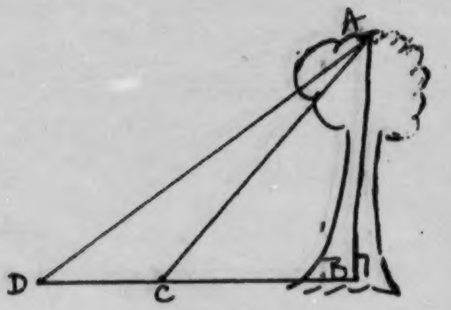
In $\triangle ADB$, $\tan 57.23^\circ = \frac{AB}{DB}$

$$\tan 57.23^\circ = \frac{AB}{70.13}$$

$$108.94 \text{ meters} = AB$$

Exercises 6.5

(1) A surveyor wishes to determine the height AB, of a tree. From C the angle of elevation to A is 43.7° and from point D the angle of elevation to A is 20.4°. If points DC and B are collinear and CD = 10 meters, how high is the tree.



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* $\measuredangle ADB$ is the angle of elevation of the top of the cliff from D.

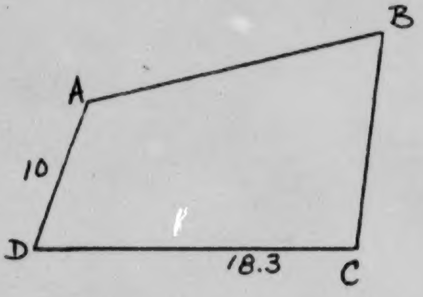
(2) Arthur Artcritic is standing 4 meters from a painting hanging on a wall of his favorite gallery. If the angle of elevation from his eye to the top of the painting is 12.6° and the angle of depression from his eye to the bottom of the painting is 8.7°, how long is the painting to the nearest tenth of a meter?

(3) Suppose that A. Artcritic has to turn his eyes 6.2° right and 3.6° left to sight the side edges of the painting. How wide is the painting to the nearest tenth of a meter?

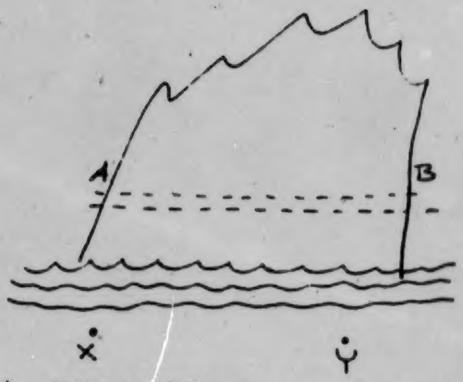
(4) Wally Weekendgolfer estimates that he has sliced his tee shot 10° from the true path between the tee and the hole. He paces off his shot and approximates that it was 175 yards. If the hole is 410 yards long, how far is he from the hole?

(5) Develop a method of determining the area of a triangle given the lengths of the three sides and find the area of a triangle whose sides are 5, 6 and 9.

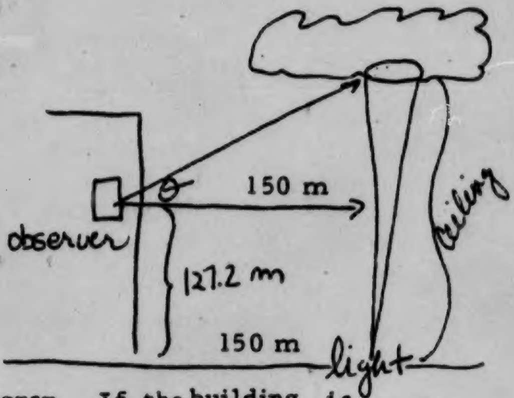
(6) Find the area of quadrilateral ABCD if AD = 10, DC = 18.3, $m\measuredangle D = 43°$, $m\measuredangle A = 107°$ and $m\measuredangle B = 84°$.



(7) In the first example in this section, how long would it take the plane to travel 719 kilometers against the wind?



(8) A surveying team wishes to determine the length of a proposed tunnel from A to B. They have instruments at X and and Y and have determined that $m\measuredangle AXY = 103.12^\circ$, $m\measuredangle BXY = 59.47^\circ$, $m\measuredangle XYB = 107.32^\circ$, $m\measuredangle XYA = 59.68^\circ$ and $m\measuredangle AYB = 73.03^\circ$. (Notice that $m\measuredangle XYB \neq m\measuredangle AYB + m\measuredangle XYA$ so X, Y, A and B are not coplanar.) If X and Y are 73.2 meters apart, what is the distance from A to B?



(9) Ground controllers at airports need to know the distance between the ground and the bottoms of the clouds. This distance is called the ceiling. One technique often used is to shine a strong light into the clouds and observe the angle of elevation θ as represented in the accompanying diagram. If the building is 150 meters from the light source at a point 127.2 meters above the level ground and measures the angle of elevation to be 65°, what is the ceiling?

(10) Make a chart at 5° intervals from 10° to 85° that could be used by the ground controllers in exercise 9.

(11) Suppose that you can swim at a rate of 2.3 kph in still water. If you are swimming across a stream whose current is .8 kph, how fast are you actually moving?

(12) Suppose that raindrops are falling at a rate of 30 kph through a wind of 23 kph. What is the measure of the angle at which the drops hit the ground (to the nearest tenth)?

Chapter 6 TEST

Directions: For (1 - 4) draw a figure. Write the formula you will be using. SHOW YOUR STEPS.

1) A diagonal of a parallelogram is 50 centimeters long and makes angles of $37^\circ 10'$ and $49^\circ 20'$, respectively, with the sides. Find the length of the shorter side of the parallelogram to the nearest centimeter.

2) Two motorboats take off from a point T at the same time. One boat travels at 10 miles per hour to a point R and the other at 20 miles per hour to a point S. If the angle STR between the two paths of travel measures 105°, find to the nearest mile, the distance between R and S after two hours.

3) In triangle ABC, angle C = 78°, side AC = 15 feet and side BC = 20 feet. Find angle A to the nearest degree.

4) Two forces of 40 pounds and 30 pounds, respectively, act on a body at the same point so that their resultant is a force of 38 pounds. Find to the nearest degree, the angle between the two original forces.

5 a) Solve for sin x: $\sqrt{1 - \sin x} = 1/2$

b) Find the value of $\cos 43^\circ 47'$ to four decimal places.

c) If x is a positive acute angle and $\cos x = \frac{\sqrt{21}}{5}$, find the numerical value of sin x.

d) In triangle ABC, b = 6, c = 10, and $m\angle A = 30$. Find the area of triangle ABC.

e) If $m\angle A = 35$, $b = 30$, and $a = 20$, then what type of triangle, if any, can be constructed?

1) a right triangle, only
2) two distinct triangles
3) one obtuse triangle, only
4) no triangle

CHAPTER 7 THE QUADRATIC AND OTHER POLYNOMIAL FUNCTIONS

In this chapter you will work with functions and equations involving powers of x, for example $f:x \rightarrow 2x^3 - 5$ and $2x^3 - 5 = 0$. Most of the content of this chapter will be directed at functions and equations of second degree, that is, powers of x no greater than two.

7.1 Polynomial Functions: Basic Definitions

A polynomial function is a function of the form $f:x \rightarrow ax^n + bx^{n-1} + cx^{n-2} + \ldots + gx + h$, n a natural number. This notation can be read "f is the function which associates with each number x, the number $ax^n + bx^{n-1} + cx^{n-2} + \ldots + gx + h$."* When $a \neq 0$ this function is named for its highest power of x and is called an n^{th} degree polynomial function. Such functions are also expressed as equations

$$f(x) = ax^n + bx^{n-1} + cx^{n-2} + \ldots gx + h.$$

The numbers represented by a, b, c, etc., are called coefficients. The terms are normally written with descending powers of the variable as shown. Polynomial functions are continuous functions, which means their graphs will contain no holes or spaces. The graphs may be obtained by plotting all (x, f(x)).

---

* In the case $f:x \rightarrow x^2 - 4$, when $x = 0$ $f:x \rightarrow -4$, this will be written $f(0) = -4$. When $x = 2$, $f(2) = 0$; $x = 3$, $f(3) = 5$, etc.

Examples:

$y = 3x^5 - 2x^2 + 7x - 9$ represents a 5th degree polynomial equation whose coefficients are 3, 0, 0, -2, 7 and -9.

$f:x \rightarrow x^4 - x^3 + 2x^2 + x - 7$ is a 4th degree polynomial function whose coefficients are 1, -1, 2, 1, and -7.

In this example

$$f(1) = (1)^4 - (1)^3 + 2(1)^2 + (1) - 7 = -4$$

$$f(2) = (2)^4 - (2)^3 + 2(2)^2 + 2 - 7 = 11$$

$$f(h) = h^4 - h^3 + 2h^2 + h - 7$$

Four polynomial functions have special names

$n = 0$	$f:x \rightarrow a$	$a \neq 0$	constant function
$n = 1$	$f:x \rightarrow ax + b$	$a \neq 0$	linear function
$n = 2$	$f:x \rightarrow ax^2+bx+c$	$a \neq 0$	quadratic function
$n = 3$	$f:x \rightarrow ax^3+bx^2+cx+d$	$a \neq 0$	cubic function

Examples:

$f:x \rightarrow \frac{x-4}{4}$	linear
$g:x \rightarrow 3(x-4)$	linear
$h:x \rightarrow 3 - x^3$	cubic
$h:t \rightarrow 100 + 60t - 16t^2$	quadratic

Exercises:

(1 - 5) State the degree of each polynomial and list its coefficients in descending order.

(1) $f:x \rightarrow 2x^3 - 6x + 4$

(2) $f:x \rightarrow 3x - 2x^2$

(3) $f:x \rightarrow 9$

(4) $f:x \rightarrow x^4 - 5x^2 - 4x + 10$

(5) $f:x \rightarrow x^3 - x - 1$

(6) For each exercise 1 - 5 above, find

(a) $f(0)$ (b) $f(1)$ (c) $f(-1)$ (d) $f(\frac{1}{2})$

(e) $f(a)$ (f) $f(x+h)$ (g) $f(-x)$

7.2 Graphing Functions

In this section we will see how the programmable calculator may assist us in graphing polynomial functions.

Given: $f:x \rightarrow x^3 - 2x^2 -5x + 6$

Graph the function.

A HP33E program

```
    PRGM
01  STO 1 1
02  2
03  -
04  RCL 1
05  X
06  5
07  -
08  RCL 1
09  X
10  6
11  +
RTN, FIX 1
```

keystroke the value of x, $\boxed{\text{R/S}}$

A TI-57 program

```
    LRN
00  STO 1
01  -
02  2
03  =
04  X
05  RCL 1
06  -
07  5
08  =
09  X
10  RCL 1
11  +
12  6
13  =
14  R/S
15  RST
LRN, RST, FIX 1
```

keystroke the value of x, $\boxed{\text{R/S}}$

To understand the program you may want to consider f:x written in factored form

$$x^3 - 2x^2 - 5x + 6$$

$$x(x^2 - 2x -5) +6$$

$$x (x (x-2) -5) + 6$$

Starting from the inside parenthesis may clarify the program.

Now we will select appropriate substitutions for x. Use the interval $[-3, 4]$[*] with a difference d of .5 between the x values. Find the values of f(x).

---

[*] The notation $[a,b]$ means values a and b are included.
(a, b) means values a and b are not included.
$[a, b)$ means a is included and b is not included

x	f(x)
-3	
-2.5	
-2	
-1.5	
-1	
-.5	
0	

x	f(x)
.5	
1	
1.5	
2	
2.5	
3	
3.5	
4	

Note: A method to increase the speed and decrease the effort needed in finding the f(x) values, would be modification of the program, such that d would automatically be added to each previous x value. This involves a loop in the program.

Modified Program for HP-33E

```
01     STO 2 (d)
02      R/S
03     STO 1 (x)
04     R/S
05 }   same as
 . }   steps 02
 . }   to 11 in
 . }   original
14 }
15     R/S
16     RCL 1
17     RCL 2
18     +
19     GTO 03
       RTN
```

Run Mode, Both programs

FIX 1

keystroke the value of d (stored in 2)

$\boxed{\text{R/S}}$

keystroke the value of x (stored in 1)

$\boxed{\text{R/S}}$

$\boxed{\text{R/S}}$

f(x) value displayed

$\boxed{\text{R/S}}$

new x displayed

$\boxed{\text{R/S}}$

new f(x) displayed

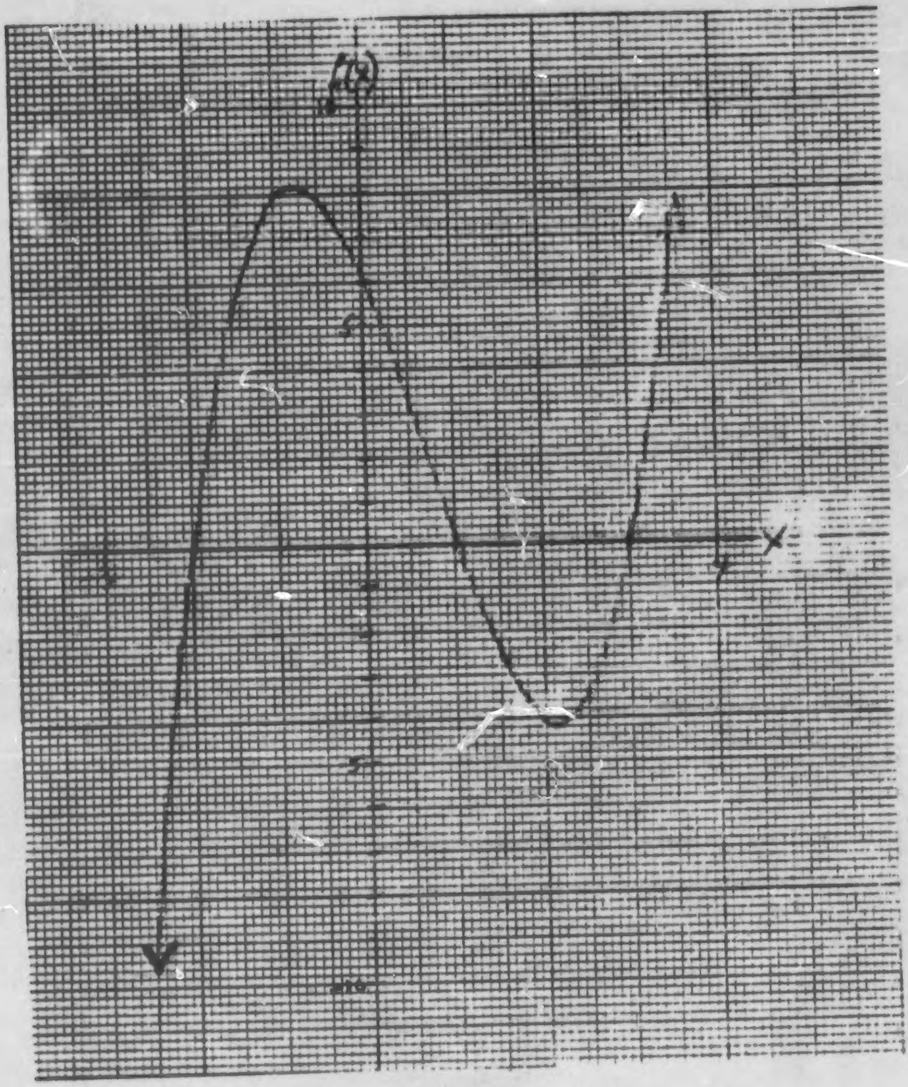
.
.
.

Modified Program for TI - 57

```
00     STO 2 (d)
01     R/S
02     Lbl 1
03     STO 1 (x)
04     R/S
05 }   same as
 . }   01 to
 . }   14 in
 . }   original
19 }
20     RCL 1
21     +
22     RCL 2
23     =
24     GTO 1
       LRN, RST
```

Finally we may sketch the graph of the function

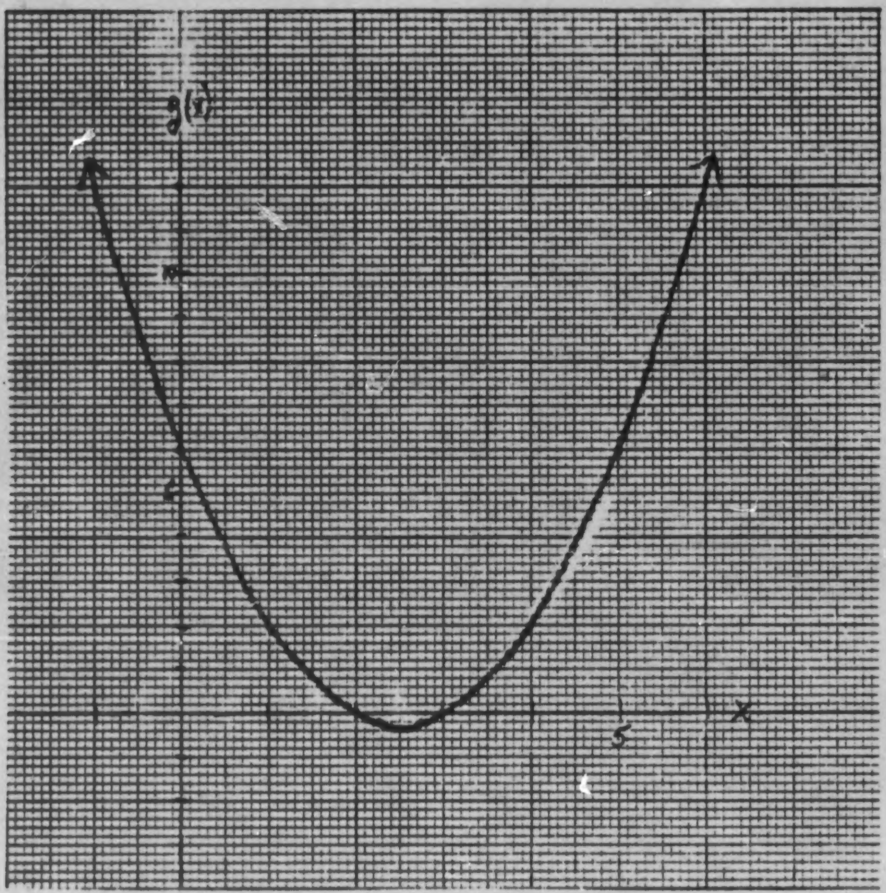
$$f:x \rightarrow x^3 - 2x^2 - 5x + 6$$



The second function that we will graph is

$$g{:}x \rightarrow x^2 - 5x + 6$$

We will use the interval $[-1, 6]$ with $d = .25$.



Exercises 7.2

Program your calculator and obtain the points necessary to graph each function. Sketch each graph using the interval and difference d as indicated.

	Function	Interval	d
(1)	$f(x) = x^2 - 3x - 10$	$[-3, 6]$	.5
(2)	$f(x) = -x^2 + 3$	$[-3, 3]$	.25
(3)	$f(x) = -x^2 + 6x - 8$	$[0, 6]$	.25
(4)	$f(x) = 2x^2 - 5x - 12$	$[-2.5, 5]$	.5
(5)	$f(x) = 2x^2 - 3x - 7$	$[-2, 3]$	.25
(6)	$f(x) = -x^3$	$[-2, 2]$	.25
(7)	$f(x) = x^3 - 3x + 3$	$[-3, 3]$	.5
(8)	$f(x) = x^3 + 2x^2 - 5x - 6$	$[-4, 3]$	.5

7.3 Graphing The Quadratic

In this section we will examine the graph of a quadratic function to determine particular f(x) values such that f(x) = k.

The smooth curve connecting all points that satisfy a quadratic function is called a parabola. Every parabola has a vertex or turning point. This point, depending on the parabola is either the maximum (highest) or minimum (lowest) point of the parabola. The parabola has symmetry about a line that passes through the vertex. It is easy to see that if the parabola was folded about this line, (called the axis of symmetry), its two halves would coincide. This axis of symmetry is the perpendicular bisector of any line segment joining two points of the parabola having the same f(x) values.

Given: $k = 180x - 16x^2$.

(a) For what value of x is k a maximum?

(b) For what values of x is k = 0?

(c) For what values of x is k = 200?

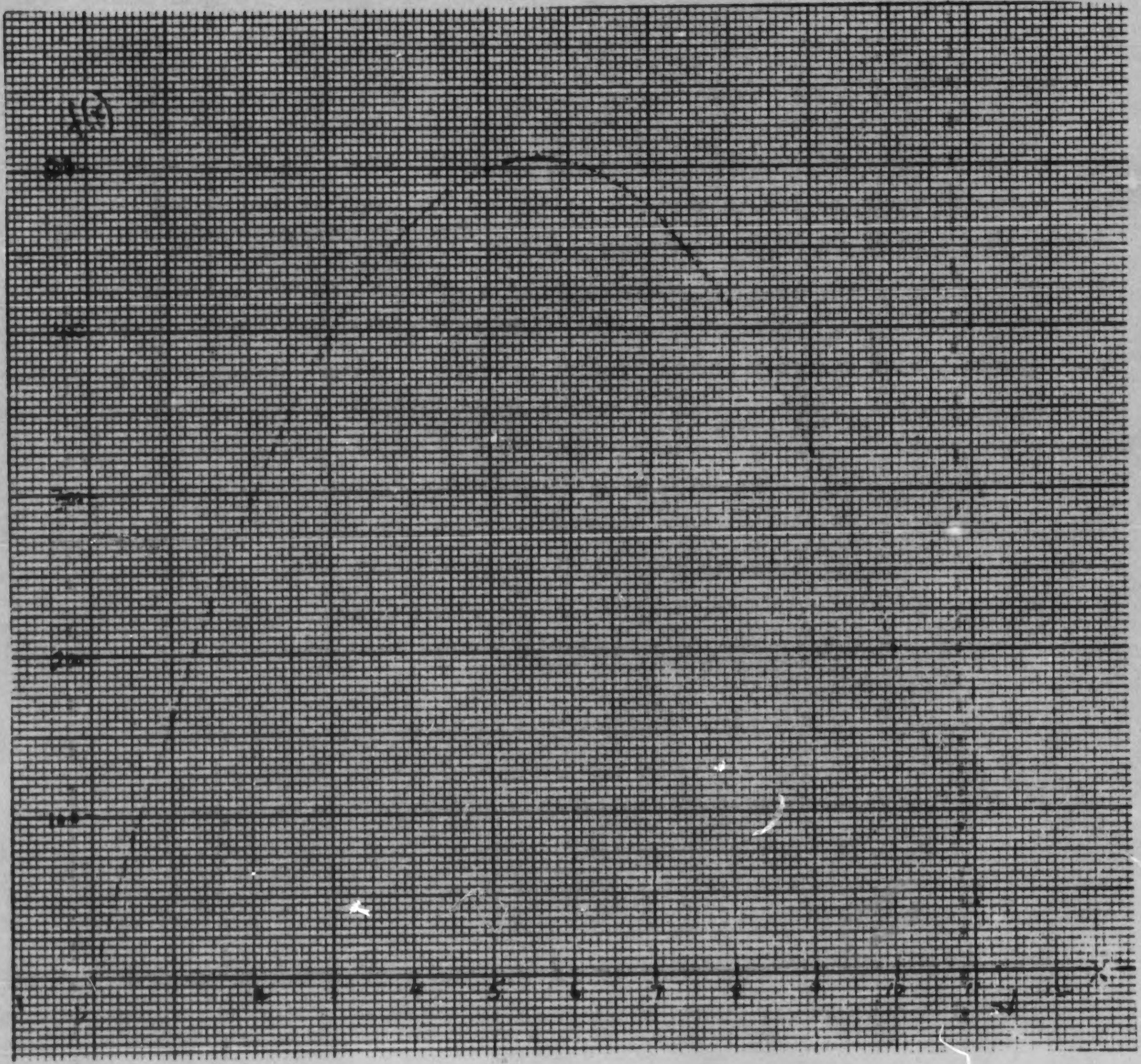
(d) For what values of x is k = 464?

To answer these questions we will graph

$$f(x) = 180x - 16x^2$$

Begin by using the interval $[-1, 12]$, d = 1. The following points should be obtained when using the calculator.

x	f(x)	x	f(x)
-1	-196	6	504
0	0	7	476
1	164	8	416
2	296	9	324
3	396	10	200
4	464	11	44
5	500	12	-144



The maximum value for f(x) seems to be located between 5 and 6. Beginning at 5.1 with d = .05 yields the following information.

x	f(x)
5.1	501.84
5.15	502.64
5.2	503.36
5.25	504
5.3	504.56
5.35	505.04
5.4	505.44

x	f(x)	
5.45	505.76	
5.5	506	
5.55	506.16	
5.6	506.24	
5.65	506.24	
5.7	506.16	f(x) is decreasing
5.75	505	

Our maximum value for x seems to lie halfway between 5.6 and 5.65, but we must be careful because of rounding off to two decimal places. Fixing our calculator to read 5 decimal places and letting d = .01 gives us

x	f(x)
5.61	506.24640
5.62	506.24960
5.63	506.24960
5.64	506.24640

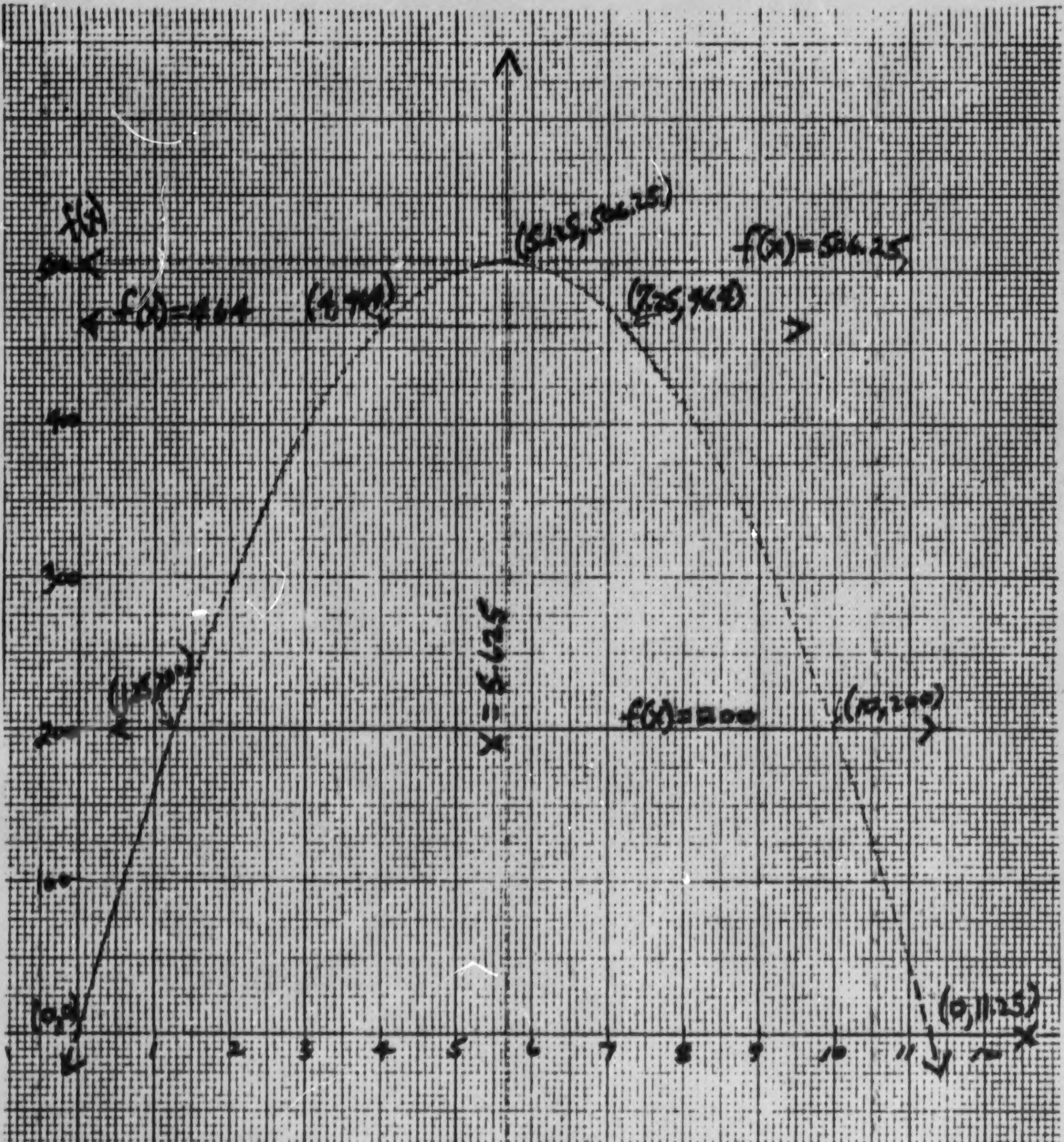
Our maximum once again seems to be halfway between 5.62 and 5.63 which is 5.625.

Using d = .001

x	f(x)
5.621	506.24974
5.622	506.24986
5.623	506.24994
5.624	506.24998
5.625	506.25 (max)
5.626	506.24998
5.627	506.24994

The maximum value for f(x) is obtained when x = 5.625. Therefore (5.625, 506.25) is the turning point or maximum in this case.

The solution to (a) is therefore 5.625 because no other value of x will yield a larger value for k, which is equal to f(x).



Solution to (b). If $k = 0$, then the solution to the problem will be found by replacing $f(x)$ in the equation $f(x) = 180x - 16x^2$ with 0.

$$180x - 16x^2 = 0$$

This clearly occurs from the graph when $x = 0$ and a second value of x between 11 and 12, (near 11.2). Using the calculator we may obtain the exact value.

x	f(x)
11.21	7.1744
11.22	5.3856
11.23	3.5936
11.24	1.7984
11.25	0

Solutions 0 and 11.25

However since 0 was 5.625 units to the left of the axis of symmetry, the second solution could have been obtained by moving 5.625 units to the right of the axis of symmetry.

$$5.625 + 5.625 = 11.25$$

(c) $180x - 16x^2 = 200$

By drawing the horizontal line $f(x) = 200$, we may estimate our solutions. However, we know from graphing that one value that satisfies this equation is 10. This value is 4.375 units to the right of the axis of symmetry. Therefore the second solution is 5.625 - 4.375 or 1.25

(d) $180x - 16x^2 = 464$

$x = 4$ (5.625 - 1.625)

or

$x = 7.25$ (5.625 + 1.625)

Exercises 7.3

(1 - 10) For each of the following functions graph the function and

(a) Find its zeros (estimate to the nearest tenth if necessary) and

(b) Find the minimum or maximum value of k in $k = ax^2 + bx + c$ such that there will be real solutions to the equation.

(1) $f(x) = x^2 + 2x$

(2) $f(x) = x^2 - 2x - 3$

(3) $f(x) = 4x^2 + 4x - 63$

(4) $f(x) = 2x^2 + 5x - 3$

(5) $f(x) = -x^2 - 1$

(6) $f(x) = -x^2 - 2x - 1$

(7) $f(x) = -x^2 + x + 5$

(8) $f(x) = -x^2 + 4x$

(9) $f(x) = x^2 - 2x - 2$

(10) $f(x) = 2x^2 - 5x + 1$

(11) You are supplied with 100' of fencing material and you desire to enclose a rectangular area using an existing wall as one of the four sides. What is the largest area you will be able to enclose? Hint: Draw a diagram and label the three sides that will be fenced in terms of x. Using the formula for area (k) of a rectangle, graph the resulting quadratic and find the maximum k.

(12) If the height above the ground of an object thrown into the air is given by $h = 100 + 128t - 16t^2$, find the maximum height attained.

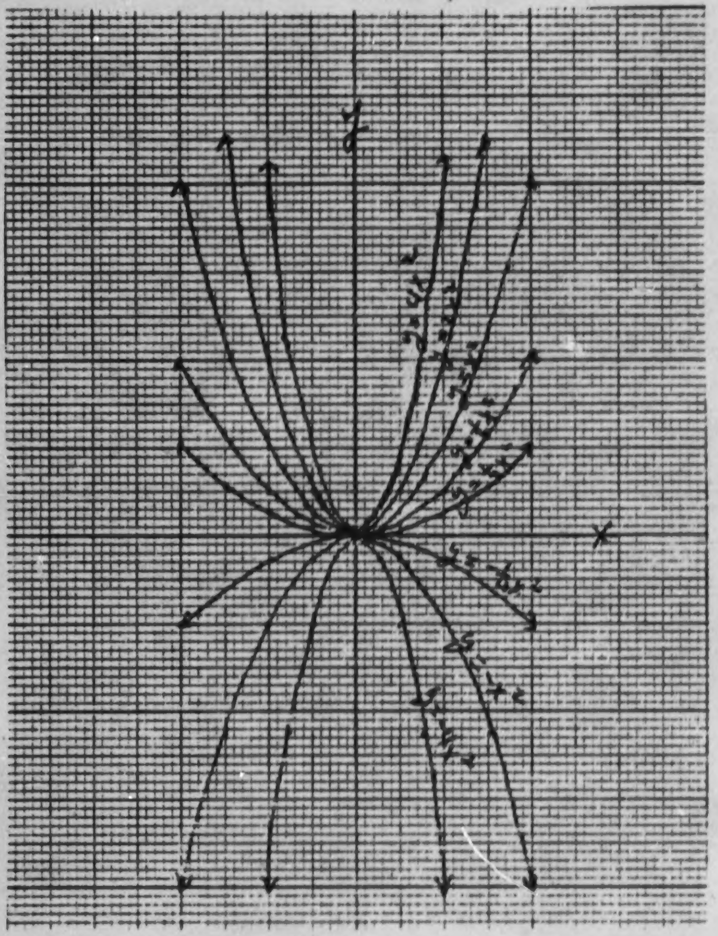
(13) Two numbers have a sum of 18. Find the numbers such that their product is a maximum.

(14) Two numbers have a sum of 20. Find the least value that the sum of their squares can attain.

7.4 The Roles of a, b, and c.

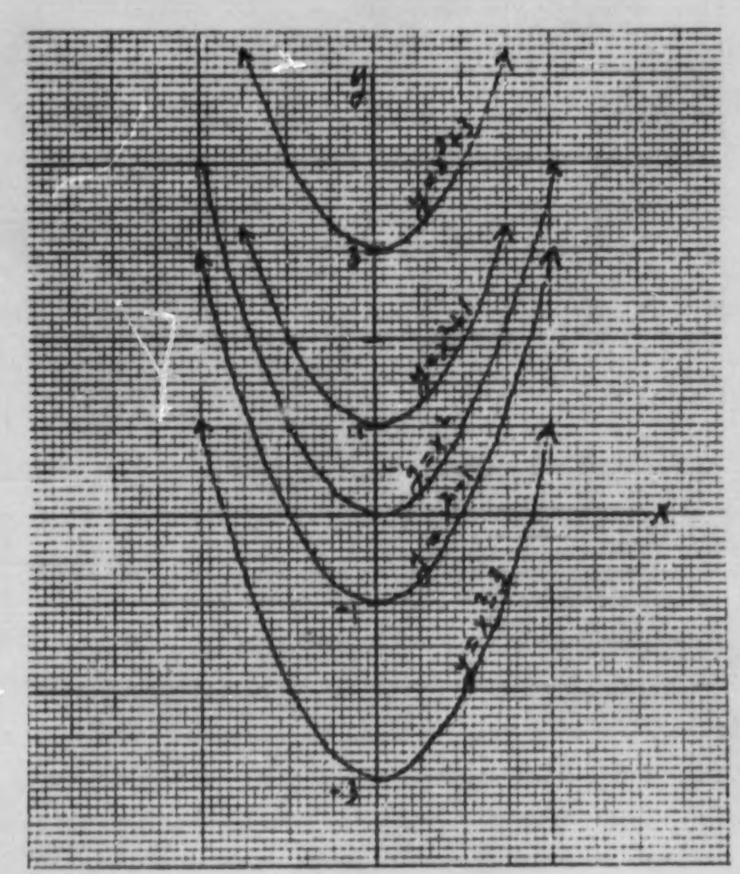
In this section discussion will include the role of the coefficients of the quadratic function $f: x \rightarrow ax^2 + bx + c$, and the rewriting of quadratic expressions by completing the square.

- The role of the coefficient a. If $b = 0$ and $c = 0$, we will sketch the graph of the equation $y = ax^2$ for the following values of a: $-4, -1, -\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4$.



Notice that when $a > 0$, the curve opens upward. As a remains positive but decreases the curve tends to flatten out. When $a < 0$, the curve opens downward and is a reflection of the graphs with the corresponding positive value of a. If $a > 0$ the turning point is a minimum. If $a < 0$ the turning point is a maximum.

- The role of the coefficients b and c. If $b = 0$ or $c = 0$, but not both. The parabola's position varies with respect to the two axes. Let $b = 0$ and $a = 1$. Graph $y = x^2 + c$ for the following values of c; -3, -1, 0, 1, 3.



All of the above parabolas have the line $x = 0$ for their axis of symmetry but the curves will be shifted vertically depending on the value of c.

if $c > 0$, there is a vertical shift upward of c units

if $c < 0$, there is a downward shift of $-c$ units

In general $y = ax^2 + c$ is obtained by shifting the graph of $y = ax^2$ up or down $|c|$ units.

In Elementary Algebra when the topic of factoring was encountered, you were introduced to special types of trinomials that when factored had two equal factors.

Examples:

$x^2 + 6x + 9 = (x + 3)(x + 3)$ or $(x + 3)^2$

$c^2 - 4c + 4 = (c - 2)(c - 2)$ or $(c-2)^2$

$4m^2 + 20m + 25 = (2m + 5)(2m + 5)$ or $(2m + 5)^2$

$p^2 + 2pq + q^2 = (p + q)(p + q)$ or $(p + q)^2$

$p^2 - 2pq + q^2 = (p - q)(p - q)$ or $(p - q)^2$

For obvious reasons this type of trinomial is called a perfect square trinomial. At times it is advantageous to have a trinomial that is a perfect square. This can be accomplished by adding a particular constant to $ax^2 + bx$ in order to form a trinomial square. This constant is $\frac{b^2}{4a}$.

$$ax^2 + bx + \frac{b^2}{4a} \text{ can be rewritten}$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) \text{ or } a\left(x + \frac{b}{2a}\right)^2$$

By this method of completing the square, any expression $ax^2 + bx + c$ can be written in the form $a(x - h)^2 + k$.

Examples:

(a) Given: $x^2 - 10x + 7$, $a = 1$, $b = -10$, therefore $\frac{b^2}{4a} = 25$. However, if 25 is added to the expression 25 must also be subtracted in order not to change the value of the original.

$$x^2 - 10x + (25) + 7 - (25)$$

$$(x - 5)^2 - 18$$

therefore $h = 5$ and $k = -18$

(b) $2x^2 + 8x - 5$, $\frac{b^2}{4a} = 8$

$$2x^2 + 8x + (8) - 5 - (8)$$

$$2(x^2 + 4x + 4) - 13$$

$$2(x + 2)^2 - 13; \qquad h = -2,\ k = -13$$

(c) What do h and k represent?

Complete the square on the function $f(x) = x^2 + x - 6$, which was graphed in section 7.3, $b = 1$, $a = 1$, $\frac{b^2}{4a} = \frac{1}{4}$

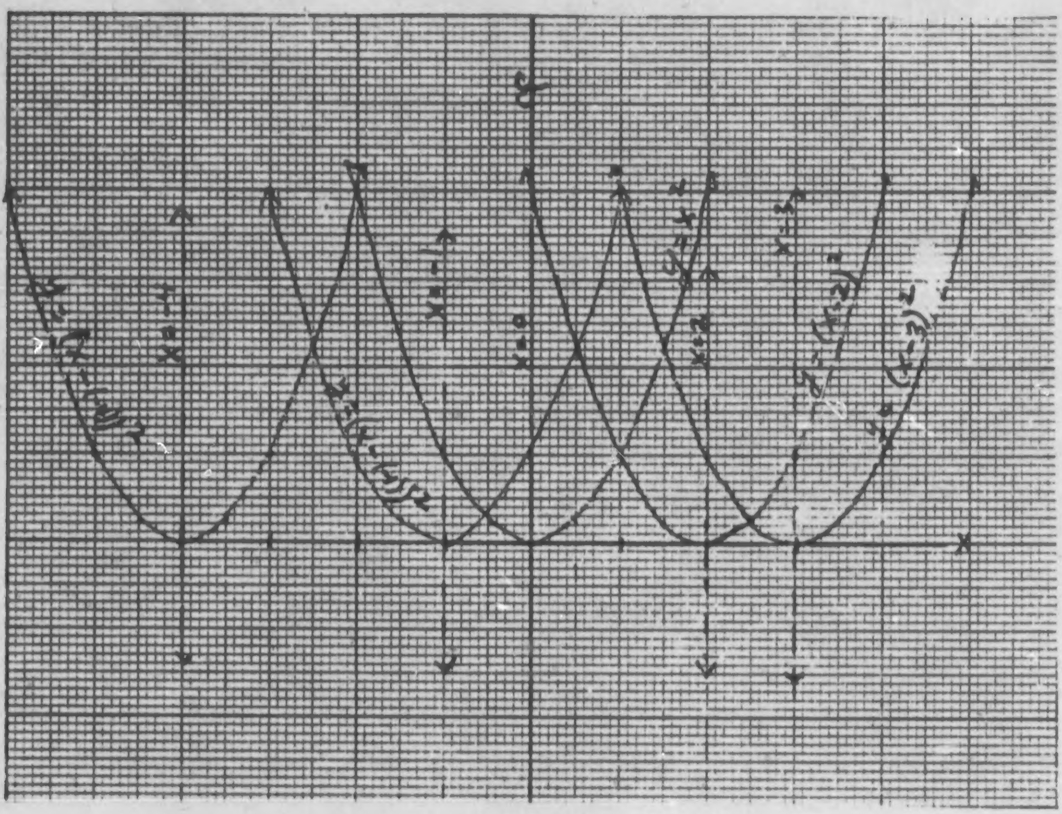
$$f(x) = x^2 + x + (\tfrac{1}{4}) - 6 - (\tfrac{1}{4})$$

$$f(x) = (x + \tfrac{1}{2})^2 - 6.25$$

$$h = -\tfrac{1}{2}, \quad k = -6.25$$

Recall that the turning point of the parabola was $(-\frac{1}{2}, -6.25)$, and the equation of its axis of symmetry the line $x = -\frac{1}{2}$. Therefore (h, k), at least in this example seems to be the turning point and $x = h$, the vertical axis of symmetry. In section 7.6 we will prove this assumption.

We will now graph $y = a(x - h)^2 + c$, for $a = 1$, $c = 0$, and the following values for h; -4, -1, 0, 2, 3.



h	Equation	Coordinates of the Turning Point	Axis of symmetry
-4	$y = (x - (-4))^2$	(-4, 0)	$x = -4$
-1	$y = (x - (-1))^2$	(-1, 0)	$x = -1$
0	$y = (x - 0)^2$	(0, 0)	$x = 0$
2	$y = (x - 2)^2$	(2, 0)	$x = 2$
3	$y = (x - 3)^2$	(3, 0)	$x = 3$

Summary: $y = a\,(x - h)^2$ differs from $y = ax^2$ only in that $y = a(x - h)^2$ has been moved $|h|$ units to the right when $h > 0$ and $|h|$ units to the left when $h < 0$. In the examples h always represents the first coordinate of the turning point. The second coordinate can be obtained by replacing x with h in the equation $y = a(x - h)^2$. $x = h$ is also the equation of the axis of symmetry.

Problem: Given $y = -2x^2 - 4x + 1$, find the highest point of the parabola without graphing.

Since $a < 0$, the parabola opens downward and has a maximum at the turning point.

$a = -2$, $b = -4$. Therefore $\frac{b^2}{4a} = \frac{16}{-8} = -2$.

By completing the square

$$y = -2x^2 - 4x + (-2) + 1 - (-2)$$

$$y = -2x^2 - 4x - 2 + 3$$

$$y = -2(x^2 + 2x + 1) + 3$$

$$y = -2(x + 1)^2 + 3 \qquad h = -1,\ k = 3$$

The vertical axis of symmetry is $x = -1$ and the turning point is $(-1, 3)$ which is the solution to the problem.

Exercises 7.4

Without graphing, find the equation of the axis of symmetry and the turning point of each of the following parabolas:

(1) $y = (x - 6)^2 + 5$

(2) $y = (x - 3)^2 - 4$

(3) $y = (x + 5)^2$

(4) $y = (x + 2)^2 - 1$

(5) $y = x^2 + 3x + 9$

(6) $y = x^2 - 14x + 49$

(7) $y = x^2 - 3x + 2.25$

(8) $y = x^2 + 5x + 6.25$

(9) $y = 4x^2 + 12x + 9$

(10) $y = 9x^2 + 30x + 25$

(11) $y = x^2 - 6x - 8$

(12) $y = x^2 - 2x + 3$

(13) $y = x^2 - x - 3$

(14) $y = x^2 + 3x - 9$

(15) $y = 2x^2 - 6x + 2$

(16) $y = 2x^2 + 4x + 3$

(17) $y = 2x^2 + 7x - 3$

(18) $y = 3x^2 - 5x - 2$

(19) $y = -3x^2 + 12x + 11$

(20) $y = -3x^2 + 6x + 2$

(21) $y = -2x^2 + 3x + 1$

(22) $y = -2x^2 - 5x + 2$

(23) Given $y = ax^2 + bx + \frac{b^2}{4a}$

Find the equation of the axis of symmetry in terms of a and b.

(24) Given: $y = ax^2 + bx + c$

Find the coordinates of the turning point in terms of a, b and c.

7.5 Solving Quadratic Equations Algebraically

In this section the zeros of a quadratic function will be found algebraically, using the methods of factoring, completing the square and the quadratic formula.

In section 7.3, by using the graphs of the parabola $f(x) = ax^2 + bx + c$ and the horizontal line $f(x) = k$ we were able to solve the equation $ax^2 + bx + c = k$. When $k = 0$, the solution is obtained by locating the point(s) of intersection (if any) of the parabola and the x axis. These first coordinates are also called roots of the equation $ax^2 + bx + c = 0$.

An algebraic method that avoids graphing would be to factor the quadratic equation, if possible, and apply the principle that if the product of factors equals zero, one of the factors equals zero.

$$\boxed{\text{If } p \cdot q = 0, \text{ then } p = 0 \text{ or } q = 0.}$$

Examples: Find the zeros of the following quadratic functions without graphing.

(a) $f(x) = x^2 - 6x$

(b) $f(x) = x^2 - 49$

(c) $f(x) = x^2 - 6x + 9$

(d) $f(x) = x^2 - 3x - 4$

(e) $f(x) = -x^2 - 4x + 1$

Solutions: To find the zeros of a function, then $f(x) = 0$

(a) $x^2 - 6x = 0$

$x(x - 6) = 0$

applying the principle

$x = 0$ or $x - 6 = 0$

$x = 0$ or $x = 6$

(b) $x^2 - 49 = 0$

$(x - 7)(x + 7) = 0$

$x - 7 = 0$ or $x + 7 = 0$

$x = 7$ or $x = -7$

(c) $x^2 + 6x + 9 = 0$ a perfect square trinomial

$(x + 3)(x + 3) = 0$

$x = -3$ only one solution. The original function is tangent to the x axis.

(d) $x^2 - 3x - 4 = 0$

$(x - 4)(x + 1) = 0$

$x - 4 = 0$ or $x + 1 = 0$

$x = 4$ or $x = -1$

(e) $-x^2 - 4x + 1 = 0$

$x^2 + 4x - 1 = 0$

not factorable.

Because not all quadratic equations are factorable, we may solve the above equation by completing the square (section 7.4). We will also use the following principles:

(i) If $x^2 = k$, then $|x| = \sqrt{k}$

and

(ii) if $|x| = m$, $m > 0$, then $x = m$ or $x = -m$

Example : (i) If $x^2 = 9$, then $|x| = \sqrt{9}$ or $|x| = 3$

(ii) If $|x| = 7$, then $x = 7$ or $x = -7$

Now we will solve this equation

$x^2 + 4x - 1 = 0$ $\qquad a = 1,\ b = 4;\ \dfrac{b^2}{4a} = \dfrac{16}{4} = 4$

completing the square

$x^2 + 4x + (4) - 1 - (4) = 0$

$x^2 + 4x + 4 - 5 = 0$

$(x + 2)^2 = 5$

$|x + 2| = \sqrt{5}$ using (i)

$x + 2 = \sqrt{5}$ or $x + 2 = -\sqrt{5}$ using (ii)

$x = -2 + \sqrt{5}$ or $x = -2 - \sqrt{5}$ solution in radical form

The two roots are irrational.

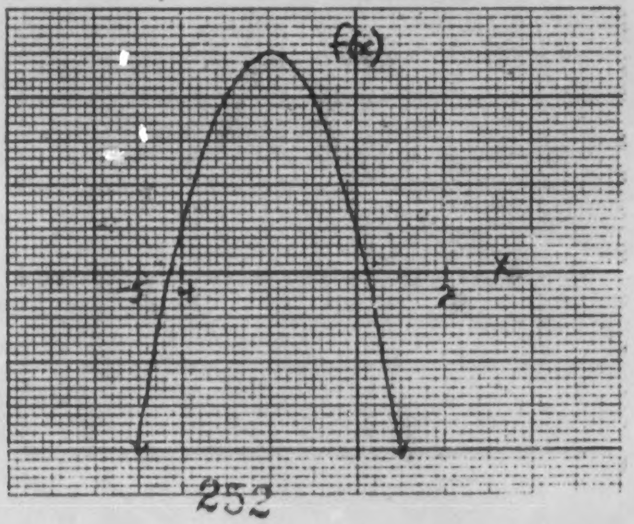
$\sqrt{5} \doteq 2.2361$

Therefore $x \doteq -2 + 2.2361$ or $x \doteq -2 - 2.2361$

$x \doteq +.2361$ or $x \doteq -4.2361$ solutions in decimal form

Explanation: $\doteq$ means equals the value to the number of decimal places specified. This is not an exact value. Checking the solutions in the original equation will not be precise.

If we attempted to solve the original function $f(x) = -x^2 - 4x + 1$, by graphing, the zeros could only at best be approximated to the nearest tenth. See graph.



We will now solve in general any quadratic equation by completing the square.

Given: $ax^2 + bx + c = 0$. Find the roots of the equation.

$$ax^2 + bx + \left(\frac{b^2}{4a}\right) + c - \left(\frac{b^2}{4a}\right) = 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} = 0$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

from (i) $$\left|x + \frac{b}{2a}\right| = \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

from (ii) $$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\boxed{x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}$$

When a, b, c represent the coefficients in descending order.

You are encouraged to remember this Quadratic Formula in order to to conserve effort when solving second degree equations.

We will now solve the equations in examples c, d and e using this formula.

(c) $x^2 + 6x + 9 = 0$

$a = 1, b = 6, c = 9$

*$b^2 - 4ac = 36 - 4(1)(9) = 0$

$x = \frac{-6 + \sqrt{0}}{2(1)}$ or $x = \frac{-6 - \sqrt{0}}{2(1)}$

since $\sqrt{0} = 0$

$x = -3$ or $x = -3$

(d) $x^2 - 3x - 4 = 0$

$a = 1, b = -3, c = -4$

$b^2 - 4ac = 9 - 4(1)(-4) = 9 -(-16) = 25$

$x = \frac{-(-3) + \sqrt{25}}{2(1)}$ or $x = \frac{-(-3) - \sqrt{25}}{2(1)}$

$x = \frac{3 + 5}{2}$ or $x = \frac{3 - 5}{2}$

$x = 4$ or $x = -1$

(e) $-x^2 - 4x + 1 = 0$

$a = -1, b = -4, c = 1$

$b^2 - 4ac = 16 - 4(-1)(1) = 16 - (-4) = 20$

$x = \frac{-(-4) + \sqrt{20}}{2(-1)}$ or $x = \frac{-(-4) - \sqrt{20}}{2(-1)}$

$x = \frac{4 + \sqrt{20}}{-2}$ or $x = \frac{4 - \sqrt{20}}{-2}$

Since $\sqrt{20} = \sqrt{4.5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$

---

* $b^2 - 4ac$ is the expression underneath the radical. This expression is called the discriminant of the equation. Its value can be very useful in determining information concerning the roots of an equation and the relative position of the graph of a function. (See exercises 2-9.)

$$x = \frac{4 + 2\sqrt{5}}{-2} \quad \text{or} \quad x = \frac{4 - 2\sqrt{5}}{-2}$$

$$x = -2 - \sqrt{5} \quad \text{or} \quad x = -2 + \sqrt{5}$$

$$\boxed{\text{If } a = b, \text{ then } a^2 = b^2}$$

This squaring principle, along with methods of solving quadratic equations algebraically may be utilized in solving radical equations.

Examples:

(f) Solve for x: $\sqrt{x - 2} + 2 = x$

It will be easier if we isolate the radical expression before applying the principle

Thus $\sqrt{x - 2} = x - 2$

and $(\sqrt{x - 2})^2 = (x - 2)^2$

$x - 2 = x^2 - 4x + 4$

$x^2 - 5x + 6 = 0$

$x = 3$ or $x = 2$

both values check in the original equation.

(g) Solve for x: $\sqrt{2x + 3} - x = 0$

$\sqrt{2x + 3} = x$

$(\sqrt{2x + 3})^2 = (x)^2$

$2x + 3 = x^2$

$x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

When we check the value $x = -1$ we have

$$\begin{array}{r|l} \sqrt{2(-1) + 3} \quad - (-1) & 0 \\ \sqrt{-2 + 3} \quad + 1 & \\ \sqrt{1} \quad + 1 & \\ 1 \quad + 1 & \\ 2 & \end{array}$$

does not check. Why not?

Since the converse of the squaring principle is not always true

i.e. $(-2)^2 = (2)^2$ but $-2 \neq 2$

a solution to the equation formed by squaring both sides is not always a solution to the original equation. If this occurs as in (g) above, the obtained root is called an extraneous root and is rejected. Thus the solution to (g) is only $x = 3$.

Exercises 7.5

(1) Why can the quadratic formula be consolidated to read

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad ?$$

(2) Refer to the solutions of examples c, d and e using the quadratic formula method.

In problem (c); $b^2 - 4ac = 0$

In problem (d) and (e); $b^2 - 4ac > 0$

What conclusions may be obtained by comparing the value of the discriminant and the roots of the equation?

(3) Without graphing, what conclusions may be made concerning a functions relationship to the x axis, if the discriminant of this function is negative?

(4) What conclusions concerning the roots of an equation may be stated if the discriminant of the equation is positive and

a) a perfect square?

b) not a perfect square?

(5) Complete the following chart (assume a, b and c, rational)

Discriminant	Roots of the Equation $ax^2 + bx + c = 0$	Graph of the function $y = ax^2 + bx + c$ in relation to x axis
If $b^2 - 4ac = 0$		
If $b^2 - 4ac < 0$		
If $b^2 - 4ac > 0$ and a perfect square		
If $b^2 - 4ac > 0$ not a perfect square		

(6) Write the discriminant of the equation $nx^2 + px + q = 0$.

(7) If the graph of the equation $y = x^2 - 6x + k$ is tangent to the x-axis, find the value of k.

(8) What are the possible values for k in the equation $y = 9x^2 + kx + 1$ if the parabola intersects the x axis at two distinct points?

(9) What are the possible values for k in the equation $y = x^2 - 3x + k$ if the parabola does not intersect the x axis?

Problems (10 - 21) Solve the quadratic equations by using the quadratic formula. If the roots are irrational, estimate these roots to the nearest tenth.

(10) $8x^2 - 6x + 1 = 0$

(11) $2x^2 - 5x - 3 = 0$

(12) $-6x^2 + 5x + 4 = 0$

(13) $-7x^2 - 4x + 3 = 0$

(14) $9x^2 = -30x - 25$

(15) $16x^2 + 8x = -1$

(16) $4x^2 + 5x - 3 = 0$

(17) $2x^2 - 8x + 1 = 0$

(18) $x^2 - 10x = -15$

(19) $3x^2 - 2 = 3x$

(20) $\dfrac{5}{x + 1} + \dfrac{x - 1}{4} = 2$

(21) $3x = 5 + \dfrac{4}{x}$

Problems (22 - 28) Solve the following quadratic equations for the trigonometric function indicated. Then find all the angles in the interval $\left[0^\circ, 360^\circ\right]$ that would satisfy the equation. When necessary estimate the angles to the nearest ten minutes.

(22) $2 \sin^2\theta - \sqrt{3} \sin\theta = 0$

(23) $2 \cos^2\theta - 3 \cos\theta - 2 = 0$

(24) $3 \tan^2\theta + \tan\theta - 4 = 0$

(25) $5 (1 - 2 \sin^2\theta) + 7 \sin\theta + 1 = 0$

(26) $\tan^2\theta + 2 \tan\theta = 4$

(27) $4 \cos^2\theta = 2 + \cos\theta$

(28) $2 \tan^2\theta = 5 \tan\theta + 1$

(29) Trace the following HP 33 program and write the results obtained at step 18 and the end.

```
      ------->   (b)                  ------->  (c)

01               CHS                  11        RCL 0
02               ENTER                12        ÷
03               2                    13        -
04               ÷                    14        f √x
05               R/S                  15        STO 2
-------------->  (a)                  16        RCL 1
06               STO 0                17        +
07               ÷                    18        R/S
08               STO 1                19        RCL 1
09               g x^2                20        RCL 2
10               R/S                  21        -
```

(30) By using the stack instead of the storage, see if you are able to shorten the program in 29.

(31) If an error signal appeared in step 14 of the program in 29, what conclusions could be made?

Solve each of the following equations for x.

(32) $\sqrt{x - 2} = 3$

(33) $\sqrt{x - 2} = \frac{x}{3}$

(34) $\sqrt{x - 2} = -x + 4$

(35) $\sqrt{x - 2} = x - 4$

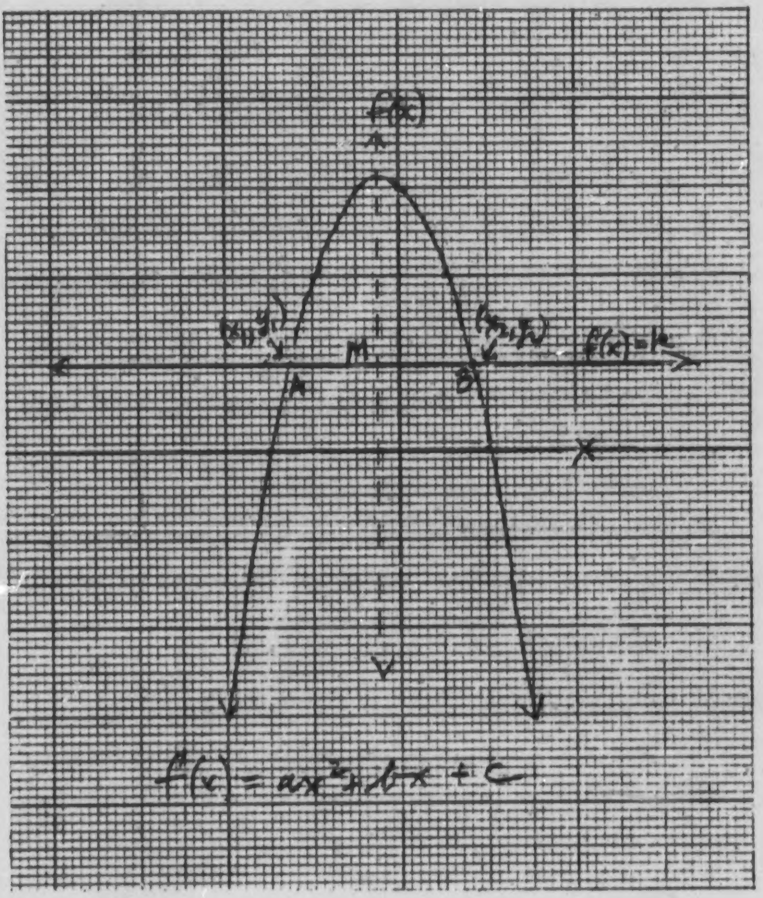
(36) $\sqrt{12 + x} = x$

(37) $2x = 5 + \sqrt{2x + 1}$

7.6 Equations for the axis of symmetry, sum and product of roots.

In this section, using the coefficients of the quadratic, we will write equations for the axis of symmetry as well as the sum and product of the roots.

Study the graph of the parabola whose equation is $f(x) = ax^2 + bx + c$.



By selecting any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the curve having the same 2nd coordinate, k, we may find their first coordinates by using the quadratic formula.

$$k = ax^2 + bx + c$$

$$0 = ax^2 + bx + (c - k)$$

$$x = \frac{-b + \sqrt{b^2 - 4a(c - k)}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4a(c - k)}}{2a}$$

Point A: $\left(\frac{-b - \sqrt{b^2 - 4a(c - k)}}{2a}, k\right)$

Point B: $\left(\frac{-b + \sqrt{b^2 - 4a(c - k)}}{2a}, k\right)$

Since by definition the axis of symmetry is the perpendicular bisector of any segment connecting two points of the parabola with the same second coordinates, we may find the coordinates of point M by using the midpoint formula.

Point M:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\begin{aligned} x_1 + x_2 &= \frac{-b - \sqrt{b^2 - 4a(c - k)}}{2a} + \frac{-b + \sqrt{b^2 - 4a(c - k)}}{2a} \\ &= \frac{-b - \sqrt{b^2 - 4a(c - k)} - b + \sqrt{b^2 - 4a(c - k)}}{2a} \\ &= \frac{-2b}{2a} \\ &= \frac{-b}{a} \end{aligned}$$

Therefore $\frac{x_1 + x_2}{2} = \frac{-b}{2a}$ and clearly $\frac{y_1 + y_2}{2} = k$

Point M: $\left(\frac{-b}{2a}, k\right)$

Since the axis of symmetry is vertical, its equation is $\boxed{x = \frac{-b}{2a}}$

Example: Find the equation of the axis of symmetry for the function $y = x^2 + x - 6$

$a = 1, \quad b = 1$ therefore $\frac{-b}{2a} = -\frac{1}{2}$

$x = -\frac{1}{2}$ is the required equation.

Referring back to graph once again -

If we had selected points A and B on the x axis, the resulting first coordinates for A and B would have been

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad b^2 - 4ac > 0$$

which represent the two roots of the equation $ax^2 + bx + c = 0$.

Now $\boxed{x_1 + x_2 = \frac{-b}{a}}$ which is the sum of these roots

and $x_1 \cdot x_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$

or

$$x_1 \cdot x_2 = \frac{\left(-b - \sqrt{b^2 - 4ac}\right)\left(-b + \sqrt{b^2 - 4ac}\right)}{4a^2}$$

using the property $(p - q)(p + q) = p^2 - q^2$

we have

$$x_1 \cdot x_2 = \frac{(-b)^2 - \left(\sqrt{b^2 - 4ac}\right)^2}{4a^2}$$

$$x_1 \cdot x_2 = \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$x_1 \cdot x_2 = \frac{4ac}{4a^2}$ $\boxed{x_1 \cdot x_2 = \frac{c}{a}}$ which is the product of the roots.

We may apply the concept of the sum and product of the roots to write quadratic equations.

Example: If the roots of an equation are $\frac{-2 + \sqrt{5}}{2}$ and $\frac{-2 - \sqrt{5}}{2}$, write the proper quadratic equation with these roots.

Solution: Since the sum of the roots is

$$\frac{-2 + \sqrt{5}}{2} + \frac{-2 - \sqrt{5}}{2} = \frac{-4}{2} = -2$$

then $\frac{-b}{a} = -2$

and since the product of the roots is

$$\left(\frac{-2 + \sqrt{5}}{2}\right)\left(\frac{-2 - \sqrt{5}}{2}\right) = \frac{(-2)^2 - (\sqrt{5})^2}{4} \qquad \frac{-1}{4}$$

then $\frac{c}{a} = -\frac{1}{4}$

The equation $ax^2 + bx + c = 0$ may be written

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

or

$$x^2 - \underbrace{(-\tfrac{b}{a})}_{\text{the sum of roots}}x + \underbrace{(\tfrac{c}{a})}_{\text{product of roots}} = 0$$

Therefore by substitution the proper equation is

$$x^2 - (-2)x + (-\frac{1}{4}) = 0$$

or

$$x^2 + 2x - \frac{1}{4} = 0$$

or

$$4x^2 + 8x - 1 = 0$$

If we investigate the inverse of a quadratic function $y = ax^2 + bx + c$ we have an equation of the type

$$x = ay^2 + by + c$$

$a > 0$, parabola opens to the right

$a < 0$, parabola opens to the left

These graphs are not functions and they have a horizontal axis of symmetry.

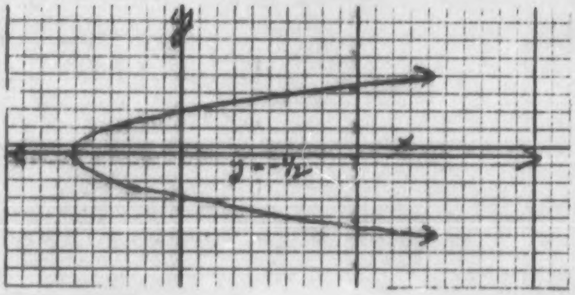
If we repeat the development at the beginning of this section we would find the equation of the axis of symmetry to be $\boxed{y = \frac{-b}{2a}}$.

Example: Find the equation of the axis of symmetry for the parabola

$$x = y^2 + y - 6$$

since $a = 1$ and $b = 1$

$y = -\frac{1}{2}$ is the equation.



Exercises 7.6

(1) Find the sum and product of the roots of the following equations

(a) $x^2 - 8x + 12 = 0$ (b) $m^2 + 6m - 4 = 0$

(c) $c^2 = 6c + 16$ (d) $3x(x-1) = 1$

(2) Given the quadratic equation $x^2 + px + q = 0$. What is the value of p if the roots of the equation are $3 + \sqrt{2}$ and $3 - \sqrt{2}$?

(3) Write the equation with integral coefficients such that the sum of the roots is $\frac{1}{3}$ and the product of the roots is $-\frac{4}{3}$.

(4) Write the quadratic equation such that the sum and product of the roots are 4 and -5 respectively.

(5) One root of $x^2 + px + 8 = 0$ is -4. Find the value of p.

(6) One root of the equation $x^2 - 7x + c = 0$ is 4. Find the other root.

(7) If the roots of the equation $ax^2 + bx + c = 0$ are opposites, find the value of b.

(8) The sum of the roots of the equation $ax^2 + 6x - 8 = 0$ is 12. Find the value of a.

(9) If the roots of the equation $x^2 + 12x + c = 0$ are equal, find the value of c.

(10) The sum of the roots of the equation $x^2 - bx + 7 = 0$ is 3. Find the value of b.

(11) Compare the sum of the roots to the product of the roots for the equation $x^2 - px + p = 0$.

(12) If the coordinates of the vertex of a parabola is (-1, -5) and the axis of symmetry passes through the point (5, -5), find the equation of the axis of symmetry.

(13) One root of the equation $x^2 = bx + 12 = 0$ is three times the second. Find two possible values for b.

(14) Write the equations whose roots are given

(a) -3 and $\frac{2}{3}$

(b) $2\sqrt{3}$ and $-2\sqrt{3}$

(c) $-\frac{5}{2} + \sqrt{3}$ and $-\frac{5}{2} - \sqrt{3}$

(d) $-\frac{4}{7}$ and $\frac{3}{7}$

(15) Given the equation $-x^2 - 4x + 1 = 0$. Check by using the sum and product of the roots to determine if $-2 + \sqrt{5}$ and $-2 - \sqrt{5}$ are the proper roots to the equation.

(16) Write the equation of the axis of symmetry for each of the following parabolas. Also find the coordinates of the vertex.

(a) $y = x^2 - 2x - 3$

(b) $y = -x^2 + 6x - 7$

(c) $y = -x^2 + 8x - 16$

(d) $x = y^2 - 16$

(e) $x = 2y^2 + 5y + 3$

(f) $x = y^2 - 5y + 4$

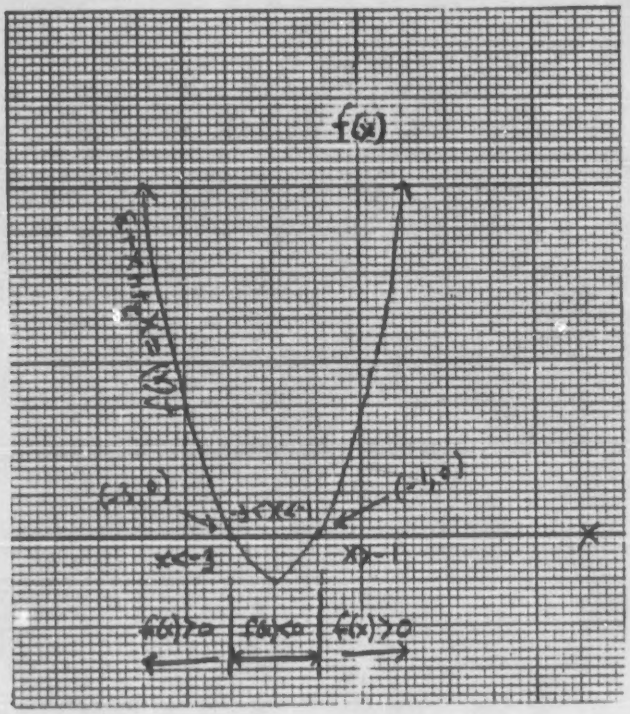
(g) $y = ax^2 + by + c$

(h) $x = ay^2 + by + c$

7.7 Inequalities

In this section we will discuss how the graphs of quadratic functions assist us in solving inequalities of the type $ax^2 + bx + c \gtrless 0$. We will also sketch graphs where $y \gtrless ax^2 + bx + c$.

Consider the parabola $f(x) = x^2 + 4x + 3$. We will refer to the points of intersection of the parabola and the x axis as cut points. Remember the x values of these cut points are the zeros of the function.



The two cut points are $(-3, 0)$ and $(-1, 0)$; these two points separate the x axis into three intervals where the second coordinate; $f(x)$ is either greater than or less than zero.

Notice, when $x < -3$, $f(x) > 0$ when $x > -1$, $f(x) > 0$ when $-3 < x < -1$, $f(x) < 0$. This information will enable us to solve problems involving this equation and any of the following symbols $>$, $\geq$, $<$, $\leq$.

Examples: Solve the following inequalities

(a) $x^2 + 4x + 3 > 0$

(b) $x^2 + 4x + 3 \geq 0$

(c) $x^2 + 4x + 3 < 0$

(d) $x^2 + 4x + 3 \leq 0$

Solutions:

(a) Since $f(x) = x^2 + 4x + 3$ for all points of the parabola, we are looking for all x such that $f(x) > 0$. This occurs when $x > -1$ or $x < -3$.

(b) The cut points are included with the solution to (a) $x \geq -1$ or $x \leq -3$.

(c) For all points between the cut points we see $f(x) < 0$. Therefore the solution is $-3 < x < -1$.

(d) $-3 \leq x \leq -1$

There are only three possible cases in determining cut points for a parabola and the x axis.

(1) No cut points such as Ⓐ

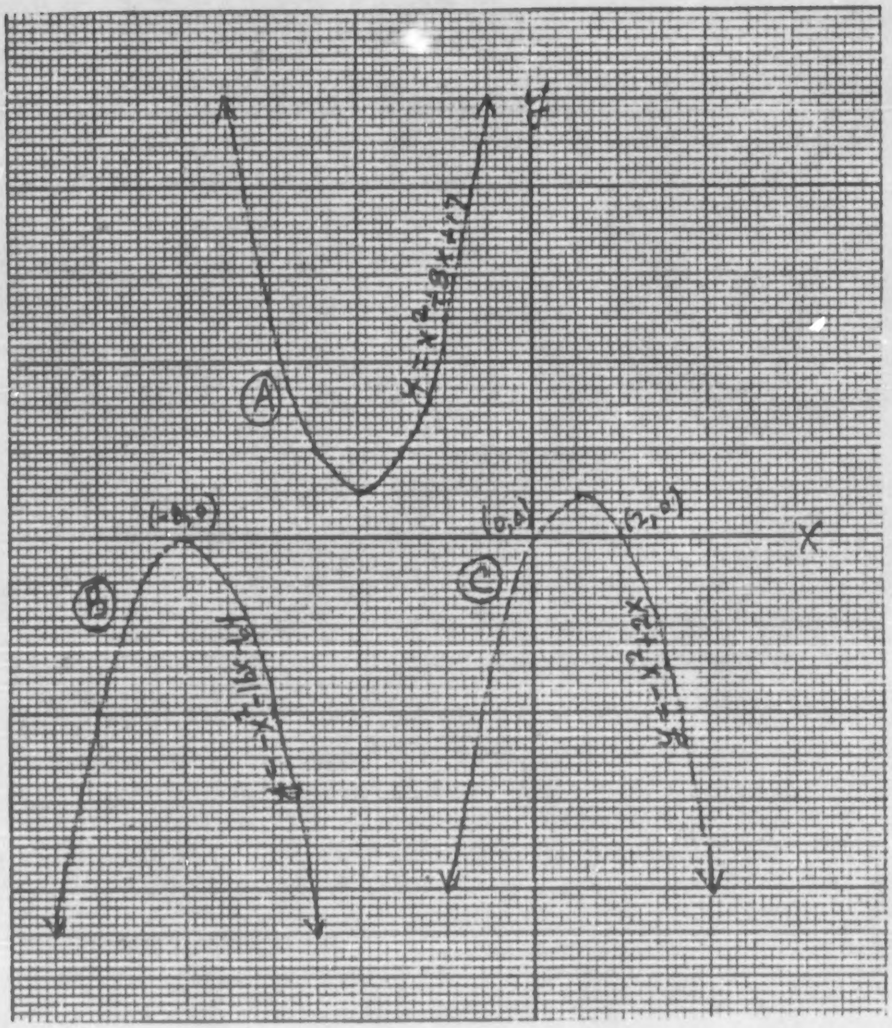
$y = x^2 + 8x + 17$ or $y = (x + 4)^2 + 1$

(2) One cut point such as Ⓑ

$y = -x^2 - 16x - 64$ or $y = -(x + 8)^2$

(3) Two cut points such as Ⓒ

$y = -x^2 + 2x$ or $y = -x(x - 2)$



For Ⓐ, there are no values for x such that $y \leq 0$.

In Ⓑ, all x except at the cut point (-8,0) we find $y < 0$.

In Ⓒ, if $-x^2 + 2 > 0$, then $0 < x < 2$; etc.

See exercises [1 - 4].

At times you may not be as concerned with the cut points but desire to compare the relationship between the coordinates.

It is obvious for example in Ⓒ, at which coordinates $y = -x^2 + 2x$ (all points on the parabola) and therefore where $y \neq -x^2 + 2x$ (all points not on the parabola). However to locate all points where, say $y > -2x^2 + 2x$ requires a testing of points not on the parabola.

The parabola acts as a borderline in solving inequalities of this type. A point not on the borderline (parabola) should be selected. If the parabola does not contain the origin, the origin would be an excellent choice.

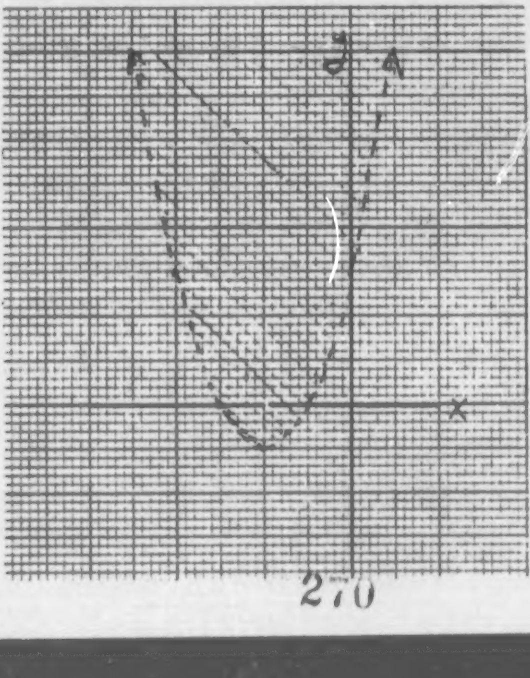
Substituting the coordinates of the selected point in the inequality will lead to a true or false statement, and thus clearly indicate if the point selected was in the correct vicinity or if it was not on the correct side of the border.

Example: Graph the inequality $y > x^2 + 4x + 3$

The graph of $y = x^2 + 4x + 3$ has already been sketched. The points on the parabola do not satisfy the inequality. Therefore draw the parabola as a dotted line, Selecting the point (0, 0) and substituting in the inequality we obtain a false statement $0 > 0^2 + 4(0) + 3$

$$0 > 3$$

Therefore the origin (0, 0) should not be in the graph of the inequality. The solution set can now be obtained by shading all points on the opposite side of the border from where the origin was located.



If the problem was to graph $y \geq x^2 + 4x + 3$, the points on the parabola would be included and therefore a solid line would be sketched.

Exercises 7.7

(1 - 4) Using the graphs Ⓐ, Ⓑ, or Ⓒ solve the following inequalities.

(1) $-x^2 + 2x \geq 0$

(2) $x^2 + 8x + 17 > 0$

(3) $-x^2 - 16x - 64 \geq 0$

(4) $-x^2 + 2x < 0$

(5 -) Sketch the graph of the inequality listed and (a) list one point that satisfies the inequality (b) list one point that doesn't satisfy the inequality.

(5) $y > x^2 + 8x + 17$

(6) $y < -x^2 + 2x$

(7) $y < x^2 + 1$

(8) $y < -x^2 + 3$

(9) Sketch the graphs of (7) and (8) on the same axis and by shading, indicate all points that satisfy both inequalities simultaneously.

Chapter 7 TEST

1) One root of the equation $3x^2 - 10x + k = 0$ is 1/3. Find the value of k.

2) The equation $x + \sqrt{x^2 + 3} = 3x$ has:

   (1) both +1 and -1 as its roots
   (2) +1 as its only root
   (3) -1 as its only root
   (4) neither +1 nor -1 as its roots

3) In the equation $x^2 - 12x + k = 0$, the sum of the roots exceeds the product of the roots by 2. The value of k is

   (1) -10 (2) 10 (3) -14 (4) 14

4) It is required that the roots of the equation $2x^2 + kx + 4 = 0$ be real numbers. A value of k which will satisfy this requirement is

   (1) 0 (2) 6 (3) -4 (4) 4

5) A value of x which satisfies the equation $\sin^2 x - 4 \sin x + 3 = 0$ is

   (1) $\pi/2$ (2) π (3) $3\pi/2$ (4) 2π

6) The equation $\sqrt{x^2 - 6x} + 4 = 0$ has

   (1) 8 and -2 as its roots (2) 8 as its only root
   (3) -2 as its only root (4) no roots

7) Find the positive value of k such that the following equation will have equal roots: $x^2 + kx + 1 = 0$

8) Find the value of x greater than 90° and less than 180° which satisfies the equation $2 \sin^2 x - \sqrt{3} \sin x = 0$

9) The sum and product of the roots of a quadratic equation are -5 and 4, respectively. An equation satisfying these conditions is

(1) $x^2 - 5x - 4 = 0$

(2) $x^2 - 4x - 5 = 0$

(3) $x^2 + 5x + 4 = 0$

(4) $x^2 + 4x + 5 = 0$

10) What is an equation of the axis of symmetry of the graph of $y = 2x^2 - x - 5$?

(1) $x = 1/4$ (2) $x = 4$ (3) $y = 1/4$ (4) $y = 4$

11) a) Find the the nearest tenth the roots of the equation $3x^2 - 4x - 2 = 0$.

b) If, in part a, $x = \cos\theta$, determine the quadrant(s) in which angle θ lies.

12) a) In the following equation, solve for tan x to the nearest tenth.

$$\tan^2 x + 2\tan x = 4$$

b) Using the results obtained in part a), find to the nearest degree the value of x in the second quadrant for which $\tan^2 x + 2\tan x = 4$.

13) a) Draw the graph of the equation $y = x^2 - 2x - 2$

b) Using the graph made in answer to part a, estimate to the nearest tenth the values of x which satisfy the equation $x^2 - 2x - 2 = 0$.

c) Using the graph made in answer to part a), find the minimum value of k for which the roots of the equation $x^2 - 2x - 2 = k$ are real.

14) a) Graph $\{(x, y) \mid y \leq 4 - x^2$ and $y > 2x + 1\}$ on the same set of axes and indicate the solution set.

b) From the graph drawn in part a), find the coordinates of a point in the solution set of $\{(x, y) \quad y \leq 4 - x^2$ and $y > 2x + 1\}$

CHAPTER 8 SEQUENCES AND SERIES

Until now you have utilized only a small fraction of the power of your calculator. In this chapter you will extend that power in important ways. Most important you will use the calculator as a decision-maker. This is the calculator function that leads some people to humanize calculators as "machines that think "; you will see, however, that the thinking will be yours, the programmer.

At the same time you will study lists of numbers called sequences and develop some important ideas that relate to two special kinds of sequences, those called arithmetic and geometric.

8.1 What are sequences?

What is a sequence and what is a series? We will answer these questions by examining an interesting program, one that will use a feature of your calculator we have not used before.

Enter the following program in your calculator:

HP-33E

```
PRGM
01      PAUSE
02      PAUSE
03      3
04      +
05      GTO 01
 RUN, RTN
 FIX 0
```

TI-57

```
LRN
00          PAUSE
01          PAUSE
02          PAUSE
03          +
04          3
05          =
06          RST
        LRN, RST
```

Now let us see what this program does. Key 5 into your display and press $\boxed{\text{R/S}}$.

What you see displayed is called a SEQUENCE, a list of specified numbers. To stop the program press $\boxed{\text{R/S}}$ again: you may have to hold it down for a second or two. You could start again from where you stopped by rekeying $\boxed{\text{R/S}}$, but instead reset your calculator by keying $\boxed{\text{RTN}}$ (HP 33E) or $\boxed{\text{RST}}$ (TI-58), and restart by keying in 5.

Rerun your program, checking your first ten numbers against the following list

5, 8, 11, 14, 17, 20, 23, 26, 29, 32

What we have listed is a sequence with ten TERMS. The fourth term is 14. The first term, which we will designate $\underline{a}$, is 5. The last term (in this case the tenth) which we will designate ℓ (for last) is 32. This is an ARITHMETIC SEQUENCE or ARITHMETIC PROGRESSION because there is a common difference $d = 3$ between terms. What we have listed is a FINITE SEQUENCE because there are a specified finite number of terms, 10. If you think of your calculator as running forever, you can think of the sequence never stopped as an INFINITE SEQUENCE. This infinite sequence could be displayed as:

$$\{5, 8, 11, 14, \ldots\} .$$

the three dots at the end meaning "continuing in the same pattern." The finite sequence could also have been displayed in this form as:

$$\{5, 8, 11, \ldots, 32\} .$$

Note that in any case where the dots (ellipsis) are used, enough terms must be listed to specify the pattern.

Exercise Set 8.1

1. Run the program of this section for a time. Stop it ($\boxed{\text{R/S}}$) when you get a number over 100. List what you believe will be the next five numbers to appear. Check by restarting the calculator ($\boxed{\text{R/S}}$). If you made a mistake, try again for the next five numbers until you are successful.
2. What characteristic of the sequence gave you your answer in (1)?

3. Form a new sequence generating program by substituting for two steps of the program on page 8.1 - 2:

HP-33E		TI - 57	
03	2	06	x
04	x	07	2

You can do this by keying over the old steps or by reprogramming entirely. This time start your program by keying

3

$\boxed{\text{R/S}}$

Record the first five terms of this sequence.

4. The sequence you generated in exercise (3) is called a GEOMETRIC SEQUENCE or GEOMETRIC PROGRESSION because it has a common multiplier, denoted $\underline{r}$ (for rate). Find $\underline{r}$ for your sequence. (Check by dividing any term by the preceeding term.)

5. What are the next three terms after 192 in your geometric sequence? Answer without your calculator, then check by using it.

6. What characteristic of the geometric sequence gave you your answer in (5)?

7. What is $\underline{a}$, the first term of your geometric sequence?

8. What is $\underline{l}$, the last (5th) term of your original geometric sequence?

9. Is an arithmetic sequence also a geometric sequence? Check by looking for a constant $\underline{r}$ in the example on page 8.1 - 3.

10. Is a geometric sequence also an arithmetic sequence? Check by looking for a constant difference $\underline{d}$ in your answer to exercise (3).

11. Reprogram your calculator as follows:

HP-33E	TI-5
PRGM	LRN
PAUSE	PAUSE
PAUSE	PAUSE
2	x
x	2
3	-
-	3
GTO 01	=
RUN, RTN	RST
FIX 0	LRN, RST

Run this program starting with:

4

$\boxed{\text{R/S}}$

Record the first five terms.

12. Is your sequence in exercise (11) arithmetic, geometric, or neither?

13. Try your program with different starting numbers. (Be sure to key RTN or RST first.) Record the first five terms in each case.

(a) 6

(b) 2

(c) 3.

14. The sequence in exercise (13(c)) is called a constant sequence. Why?

15. Identify each sequence in exercise (13) as arithmetic, geometric, or neither.

16. Think of some other rules for generating sequences.

8.2 Series: Sums of Sequences

In the text of section 8.1 and in exercises (3), (11), and (13c) for that section we generated four sequences with our calculators. We will label these sequences for convenience:

$$A = \{5, 8, 11, 14, 17, 20, \ldots\}$$

$$G = \{3, 6, 12, 24, 48, 96, \ldots\}$$

$$B = \{4, 5, 7, 11, 19, 35, \ldots\}$$

$$C = \{3, 3, 3, 3, 3, 3, \ldots\}$$

We will use these examples of sequences to form series.

SERIES are sequences which are sums of other sequences. For example we may form a series from sequence A in the following way:

1st term: 5
2nd term: $5 + 8 = 13$
3rd term: $5 + 8 + 11 = 24$
4th term: $5 + 8 + 11 + 14 = 38$

and the series associated with sequence A is

$$S(A) = \{5, 13, 24, 38, 55, 75, \ldots\}$$

Clearly you could shorten the calculations (always important for either efficiency or laziness) to adding only the next term:

1st term: 5
↓
2nd term: $5 + 8 = 13$
↙
3rd term: $13 + 11 = 24$
↙
4th term: $24 + 14 = 38$
⋮

We are often interested in a particular term of a series called a partial sum. We will denote partial sums by indicating the number of the term as a subscript. Thus

$$S_2(A) = 13$$

$$S_5(A) = 55$$

Similarly you can check by addition to see that

$$S_3(G) = 21$$

How can we use calculators to find the terms of a series? If you thought about that for a few minutes you would see that you could do this by modifying your programs for sequences in order to accumulate the sums of terms in the following way:

1. Place first sequence term in accumulator to form S_1
2. Calculate next term of sequence.
3. Add this to accumulator
4. Return to step 2.

Note that the LOOP we form by returning from step (4) to step (2) is just like the loop we formed in calculating our original sequence. There is one catch, however; we must retain two separate terms, the term of our sequence (needed to generate the next term) and the term of our series (also needed to generate the next term). We can do this by setting up two storage locations R_1 for the sequence term

R_2 for the series term

Now let us set up our program for summing the sequence 5, 8, 11, 14, . . .

HP-33E	TI-57	NOTES
PRGM	LRN	
01 STO 1 02 STO 2	00 STO 1 01 STO 2	Store the initial term in R_1 and R_2
03 PAUSE 04 PAUSE	02 Lbl 1 03 PAUSE 04 PAUSE	Display the sum so far
05 RCL 1 06 3 07 +	05 RCL 1 06 + 07 3 08 =	Calculate the next sequence term
08 STO 1	09 STO 1	Store it in R_1
09 STO + 2	10 SUM 2	Add it to R_2
10 RCL 2	11 RCL 2	Recall the new series (sum) term
11 GTO 03	12 GTO 1	Loop to display
RUN, RTN FIX 0	LRN, RST	

Now the series program may be run by keying

5

R/S

Run the program and check your results against the terms on page 8.2 - 1.

Several important new calculating concepts are introduced here. The first is called REGISTER ARITHMETIC. You can substitute a new value for an old one by keying

STO n

But register arithmetic allows you to modify what is in a register. For example, if the number 5 is stored in R_2 and we key

HP-33E		TI-57
8		8
STO + 2	or	SUM 2

The result will be 13 stored in R_2, 8 remaining in the display. By similar techniques registers may have their contents modified by subtraction, multiplication, and division.

Much more important is what we have designated LOOPING. One of the greatest powers of the programmable calculator (and the computer) is this ability to repeat operations over and over at the programmer's instruction.

Some calculators allow looping simply by a GTO instruction followed by a specific step number. For example, the HP-33E instruction

GTO 05

will send the calculator to instruction number 05, to carry on from there. Other calculators like the TI-57, require a label to be inserted where the program is to be sent. In the program on page 8.2 - 4, for example, the label 1 is inserted in step 02. The program is directed to that step in step 12.

Exercises 8.2

1 - 4 (HP-33E), 5 - 8 (TI-57)

For each of the following sequences of steps, tell what is in R_1 and what is in the display.

(1)
5
STO 1
4
STO + 1

(2)
5
STO 1
4
STO - 1

(3)
5
STO 1
4
STO x 1

(4)
5
STO 1
4
STO ÷ 1

(5)
5
STO 01

4
SUM 1

(6)
5
STO 1

4
INV
SUM 1

(7)
5
STO 1

4
PRD 1

(8)
5
STO 1

4
INV
PRD 1

(9) Program your calculator to display S(G). (Recall that G is a geometric sequence with r = 2.) Record the first six terms.

(10) Program your calculator to display S(B). (You may wish to refer to section 8.1 to recall the generating rule.) Record the first six terms.

(11) Program your calculator to display S(C). (Recall that this program is the same as that of exercise (10) but with a different value of a.) Record the first six terms.

(12) Is the series in exercise (11) arithmetic, geometric or neither? Why?

(13) Calculate $S_3(A)$.

(14) Calculate $S_3(G)$

(15) Calculate $S_3(B)$

(16) Calculate $S_3(C)$

8.3 Beating your calculator: Arithmetic Progression Formulas

Since you are the controller of your calculator, you ought to be able to beat it at will. For example, if you want to win at the simplest level, ask your machine the sum of 2 + 2. Before you have time to touch the keys, you know and can "beat" the calculator with the answer, 4.

But here we seek to beat the calculator in a quite different sense. We want to analyze arithmetic and geometric sequences so that we can find information like the thirtieth term in a particular sequence or the sum of those thirty terms quicker than the brute force methods of calculation of sections 8.1 and 8.2.

To do this we will develop some formulas. In doing so the following notation is used:

a - first term of a sequence

d - common difference of an arithmetic sequence

ℓ - last term of a finite sequence

n - number of terms of a sequence

$S_n(X)$ - sum of the first n terms of sequence X.

First we examine arithmetic progressions, first term a, common difference d.

1st term: a

2nd term: $a + d$

3rd term: $a + d + d = a + 2d$

4th term: $a + d + d + d = a + 3d$

⋮

Do you see the general pattern of the terms? Notice how the coefficient of d is related to the number of the term. This suggests the general term

$$n^{th} \text{ term:} \qquad a + (n-1)d$$

This formula is usually stated for a finite arithmetic sequence of n terms:

$$\boxed{\ell = a + (n-1)d}$$

Example 8.3.1 Find the 50th term for the arithmetic progression $\{5, 8, 11, 14, \ldots\}$

Solution: Here $a = 5$, $d = 3$, and $n = 50$, so

$$l = 5 + (50-1)3 = 5 + 49 \cdot 3 = 5 + 147 = 152.$$

Now we consider the terms of the related series. Recall that we must add the terms of the corresponding sequence. First we look at the sequence and series together:

TERM	SEQUENCE	SERIES
1	a	a
2	$a + d$	$2a + d$
3	$a + 2d$	$3a + 3d$
4	$a + 3d$	$4a + 6d$
5	$a + 4d$	$5a + 10d$
.	.	.
.	.	.
.	.	.
n	$a + (n-1)d$	?

It appears that the sum is of the form $S_n = na + ?\,d$, but replacing the ? is a problem. We employ a great trick.

First recall that the n^{th} term is also ℓ. Thus we can say that the sum of n terms is:

$$S_n = a + a{+}d + a{+}2d + \ldots + \ell - 2d + \ell{-}d + \ell$$

Now the trick. We write this formula twice, the second time reversing the order.

$$S_n = a + a{+}d + a{+}2d + \ldots + \ell{-}2d + \ell{-}d + \ell$$

$$S_n = \ell + \ell{-}d + \ell{-}2d + \ldots + a{+}2d + a{+}d + a$$

and add (1):

$$2S_n = a{+}\ell + a{+}\ell + a{+}\ell + \ldots + a{+}\ell + a{+}\ell + a{+}\ell$$

Now since there were n terms in Sn we can write this as:

$$2S_n = n(a + \ell)$$

from which:

$$\boxed{S_n = \frac{n}{2}\,(a + \ell)}$$

Example 8.3.2 Find the sum of 50 terms of the arithmetic progression

$$\{5,\ 8,\ 11,\ 14,\ \ldots\}\ .$$

Solution: $a = 5$, $\ell = 152$ (from example 8.3.1), and $n = 50$.

$$S_n = \frac{50}{2}\,(5 + 152) = 25 \cdot 157 = 3925$$

In example 8.3.2, we needed to know ℓ in order to use the formula for S. But we know a formula for ℓ. We can substitute it in our formula for S.

Since

$$\ell = a + (n-1)d \quad \text{and} \quad S = \frac{n}{2}\,(a+\ell)$$

we have

$$S_n = \frac{n}{2}[a + a + (n-1)d] = \frac{n}{2}[2a + (n-1)d]$$

We can multiply out the brackets to get

$$S_n = na + \frac{n(n-1)}{2}d \qquad (*)$$

which relates to the values of the table on 8.3 - 2, but the formula is more often given in the earlier form

$$\boxed{S = \frac{n}{2}[2a + (n-1)d]}$$

Example 8.3 - 3. Find the sum of ten terms of the arithmetic progression $\{-7, -5, -3, -1, \ldots\}$

Solution: $a = -7$, $d = 2$, $n = 10$.

$$S = \frac{10}{2}[2(-7) + (10 - 1)2] = 5[-14 + 18] = 5 \cdot 4 = 20$$

ARITHMETIC PROGRESSION FORMULA SUMMARY

$$\ell = a + (n-1)d$$

$$S_n = \frac{n}{2}(a + \ell)$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Exercises 8.3

Given the arithmetic progressions

$P = \{1, 5, 9, \ldots\}$

$Q = \{-19, -16, -13, \ldots\}$

$R = \{12, 10, 8, \ldots\}$

---

* This formula will be referred to later in Exercise 14.

$S = \{0, 0.75, 1.50, \ldots\}$

$T = \{1, 2, 3, \ldots\}$

use formulas developed in this section to answer the following:

1) Find the tenth term of each sequence.

2) Using (1) find the sum of the first ten terms for each sequence.

3) Find S_{15} (S)
4) Find S_{20}(Q)

5) Find S_{30}(P)
6) Find S_{30}(R)

7) What is the n^{th} term of T?

8) Notice the difference in sign of the answers in exercises 4) and 6) from the initial terms of the sequences added. How do you explain this?

9) Find S_n(T). Your answer will be in terms of n.

10) Now it is time to see if you can beat your calculator. Using the first program of section 8. 1, time yourself as you find the 30th term in the sequence $\{5, 8, 11, 14, \ldots\}$. (Note: you will have to count as the calculator displays terms until you come to the 30th.) Record this 30th term as well.

11) Now time yourself as you calculate the 30th term of $\{5, 8, 11, 14, \ldots\}$ using an arithmetic progression formula. Check your answer against that of exercise 10; the answers should be the same (If not, did you forget to count the starting term, 5?) Why are your times different?

12) Repeat exercises 10) and 11) but using the first program of section 8.2 to find the sum of the first 30 terms of $\{5, 8, 11, 14, \ldots\}$.

13) In a famous story the young mathematician, Frederick Gauss, answered a school question designed to keep students busy, in a matter of seconds. The exercise was to find the sum of the first 100 counting numbers. Gauss did not have our formulas handy, but he may have used the trick of our proof on page 8. 3 - 3. Solve Gauss's problem by this method. Start with

$$S = 1 + 2 + 3 + \dots 98 + 99 + 100.$$

14) Recall that the terms of the series in the table on page 8. 3 - 2 were $\{a,\ 2a+d,\ 3a + 3d,\ 4a + 6d,\ 5a + 10d,\ \dots\}$. Compare the terms of this series with the formula marked (*) on page 8. 3 - 4.

8.4 Beating your calculator: Geometric Progression Formulas

In this section we seek formulas for the n^{th} term and the sum of n terms of a geometric progression. Here we will be concerned with $\underline{r}$, the common multiplier or rate, rather than $\underline{d}$, the difference between terms. Otherwise the notation -- $\underline{a}$, $\underline{n}$, $\underline{l}$, $S_n(X)$ -- will remain the same.

First we examine the terms of a geometric progression:

1^{st} term: a
2^{nd} term: ar
3^{rd} term: $ar \cdot r = ar^2$
4^{th} term: $ar^2 \cdot r = ar^3$
⋮

Here the pattern should be clear.

n^{th} term (l) ar^{n-1}

and we state this formula

$$\boxed{\ell = ar^{n-1}}$$

Example 8.4.1 Find the fifth term of the geometric progression $\{7, 14, 28, \ldots\}$.

Solution: Most thinking people would merely double twice more but the formula solution is

$$\ell = 7 \cdot 2^4 = 7 \cdot 16 = 112$$

Example 8.4.2 Find the tenth term of the sequence $12, -6, 3, -\frac{3}{2}, \ldots$

Solution: Here $a = 12$, $r = -\frac{1}{2}$, $n = 10$

$$\ell = 12\left(-\frac{1}{2}\right)^9$$

Check with your calculator* that this is $\ell = -.0234375$ and that this is also $\ell = -3/128$

Now let us attempt to derive a formula for $S_n(X)$ where X is a geometric progression.

TERM	SEQUENCE	SERIES
1	a	a
2	ar	$a + ar$
3	ar^2	$a + ar + ar^2$
4	ar^3	$a + ar + ar^2 + ar^3$
.	.	.
.	.	.
.	.	.

We should have expected something messy like this because we are multiplying in the sequences and adding in the series. At any rate writing our formula as

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

is no shortcut. It is merely adding all of the sequence terms. We could have done that without a formula.

But now is the time for another trick. (In stage plays this is called a deus ex machina - a mechanical contrivance or godly intervention.) Here is ours: we multiply each term by r, giving:

---

* If your calculator will not accept a negative base in the calculation, determine the sign of the answer by observing that even numbered terms are negative, then calculate the decimal using 1/2 as base.

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$$

and subtract from the first equation to get:

$$S_n - rS_n = a - ar^n$$

the other terms having dropped out in the process.

Now we can factor the left member

$$S_n(1-r) = a - ar^n$$

and divide by $1 - r$ to obtain

$$\boxed{S_n = \frac{a - ar^n}{1 - r}}$$

Example 8.4.3 Find the $S_6(X)$ where $X = \{100, 110, 121, \ldots\}$.

Answer correct to two decimal places.

Solution: $a = 100$, $r = 1.1$, $n = 6$

$$S_6 = \frac{100 - 100(1.1)^6}{1 - 1.1} = \frac{100(1.1)^6 - 100}{0.1} \quad \text{(changing signs)}$$

Check by calculator that this simplifies to

$$S_6 = 771.56$$

In our development of arithmetic progression formulas we also had a formula for S_n involving ℓ. Noting that $\ell = ar^{n-1}$ and $r\ell = ar^n$ we can substitute this in the boxed formula above to get

$$\boxed{S_n = \frac{a - r\ell}{1 - r}}$$

In summary we have the three geometric progression formulas

$$\boxed{\begin{array}{l} \ell = ar^{n-1} \\ S_n = \dfrac{a - r\ell}{1 - r} \quad \text{or} \quad S_n = \dfrac{r\ell - a}{r - 1} \\ S_n = \dfrac{a - ar^n}{1 - r} \quad \text{or} \quad S_n = \dfrac{ar^n - a}{r - 1} \end{array}}$$

The alternate formulas are the result of changing the sign of numerator and denominator of the first formulas. They are useful to keep signs positive in exercises like example 8.4.3.

Exercises 8.4

1 - 8. Find the indicated term. Be sure to determine first what kind of sequence you have.

1) $\{50, 10, 2, \ldots\}$ 10th

2) $\{1.2, 1.08, 0.96, \ldots\}$ 41st

3) $\{1, 0.4, 0.16, \ldots\}$ 7th

4) $\{18, -6, 2, \ldots\}$ 12th

5) $\{1, \sin\theta, \sin^2\theta, \ldots\}$ 8th

6) $\{\tan\theta, 1, \text{ctn}\,\theta, \ldots\}$ 30th

7) $\{1, 2, 4, 7, 11, 16, \ldots\}$ 8th

8) $\{\frac{3}{16}, \frac{3}{4}, 3, \ldots\}$ 12th

9 - 16. Find the sum of each sequence.

9) $\{40, 20, 10, \ldots\}$ 15 terms

10) $\{1, 1/2, 1/4, \ldots\}$ 20 terms

11) $\{1, -\frac{1}{5}, \frac{1}{25}, \ldots\}$ 8 terms

12) $\{4, 2.5, 1, \ldots\}$ 12 terms

13) $\{-18, -6, -2, \ldots\}$ 13 terms

14) $\{3, 3.25, 3.5, \ldots\}$ 16 terms

15) $\{1, 2, 4, 8, \ldots, 16384\}$

16) $\{500, 250, 125, \ldots, 3.90625\}$

17) Now we test again to see if your shortcut formulas beat the calculator program. Use the program for 3 of section 8.1 to find the 20th term of the geometric progression $\{3, 6, 12, \ldots\}$. Record the time taken.

18) Find ℓ in exercise 17 by formula using your calculator but not the program. Record how long it takes you and compare with your time in exercise 17.

19) Use the procedures of exercises 17 and 18 to compare times to find the sum of 20 terms of the geometric progression $\{5, 1, 0.2, \ldots\}$.

20) Use the identity

$$1 + r + r^2 + r^3 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}\,^*$$

to convert

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

to the formula $S_n = \dfrac{a - ar^n}{1 - r}$

21) $\{1, 10, 100, 1000, \ldots\}$ is a geometric progression. What kind of sequence is the one formed by taking logs of each term? (Use your calculator to convert if necessary.)

22) $\{10, 1, 0.1, 0.01, \ldots\}$ is a geometric progression. What kind of sequence is formed by taking logs of each term?

23) Write the log equation for $\ell = ar^{n-1}$. How does this answer compare with the arithmetic progression formula for ℓ?

24) $\{1, 3, 5, 7, \ldots\}$ is an arithmetic progression. Form a new sequence by using these terms as powers of 4: $\{4^1, 4^3, \ldots]$. What kind of sequence is this?

25) Using the ideas of exercise 24, show that by using the terms of the arithmetic progression $\{a, a + d, a + 2d, \ldots\}$ as powers of x $(x > 0)$ the new sequence formed

---

* If you are good at polynomial division you can check this by dividing $1 - r \overline{)\, 1 + 0r + \ldots + 0r^{n-1} - r^n}$. You may wish to try special cases like $n = 4$ first.

$$\left\{x^{a},\ x^{a+d},\ \ldots\right\}$$

is a geometric progression.

26) How are r and d related in exercise 25?

8.5 Counting Loops

In sections 8.3 and 8.4, you should have realized that we were unfair to the calculator when we used the formula and forced the calculator to calculate terms (with long pauses between calculations.) If we had programmed the formulas into the calculator all of us but calculating prodigies would have lost the races set up in the exercises.

We have also used very rudimentary programs for our computations. After all, if we're interested in only the 30th term of a program, why display all the terms up to that one? And why ask ourselves to keep count? In this section we assign counting to the calculator and in the next we program it to calculate answers directly.

*

Counters are easy to insert into loops. Merely establish a new storage location, say R_5. Be sure that this is not used in other parts of your program. Set $R_5 = 0$ at the beginning of program operation and then insert the following steps at an appropriate place in your loop:

HP-33E	TI-57
1	1
STO + 5	SUM 5

To show how this works, here are programs that will just include this counter, displaying the result at each step:

```
      HP-33E                    TI - 57

     PRGM                         LRN
01       0                  00  0
02       STO  5             01  STO  5
03       1                  02  Lbl  1
                            03  1
04       STO + 5            04  SUM  5
05       RCL  5             05  RCL  5
06       PAUSE              06  PAUSE
07       GTO  03            07  GTO  1
         RUN,  RTN            LRN,  RST
```

Now let's insert that counter into a program to let it take over our counting of steps. We will do this by modifying our program of section 8.1 for generating sequence terms. We'll apply the technique to our old friend

$$\{2, 5, 8, \ldots\}$$

Here are the programs. We will use $R_0 = a_n$ (the n^{th} term), $R_1 = n$ (the counter)

```
        HP-33E               TI - 57          Comments

       PRGM                   LRN

01      STO    0        00    STO  0        a into Ro
02      0               01    0             set counter R1 to 0
03      STO    1        02    STO  1
04      1               03    Lbl  1
05      STO + 1         04    1
06      RCL    1        05    SUM  1
07      PAUSE           06    RCL  1
08      RCL    0
09      PAUSE
10      3
```

```
11     STO + 0                                  Increase counter
12     GTO   04        07     PAUSE             by 1 and display
       RUN, RTN        08     RCL 0

                       09     PAUSE             Display a_n
                       10     3
                       11     SUM 0             Calculate a_n+1

                       12     GTO 1             Loop
                           LRN, RST
```

Enter this program in your calculator. To run it key

2 (this is a)
R/S

Your calculator will display n then a_n. Note how much faster it runs than our earlier programs because we inserted fewer PAUSEs.

Exercises 8.5

1 - 4. Use the program of this section to find

1) a_{10} 2) a_{12} 3) a_{30}

4) Time your calculator to find how long it takes to calculate a_{40}.

5) Use your answer in exercise (4) to determine to the nearest tenth of a second how long it takes for the calculator to complete each loop.

6 - Modify the program of this section to generate the geometric progression $\{3, 6, 12, \ldots\}$ with a counter. Use it to find

6) a_{10} 7) a_{15} 8) a_{30}

9) How long does it take your calculator to calculate a_{40} for this sequence?

10) What term in this sequence is 196,608?

11) How many terms of this sequence are required to generate a term over 100,000,000?

12) Modify the program for $S_n(X)$ for $X = \{5, 8, 11, \ldots\}$ (see 8.1-2) to include a counter. Use it to calculate $S_{10}(X)$.

13) Modify the program of exercise 12 to calculate $S_n(Y)$ for $Y = \{3, 6, 12, \ldots\}$. Use it to calculate $S_{10}(Y)$.

8.6 Machines that Think

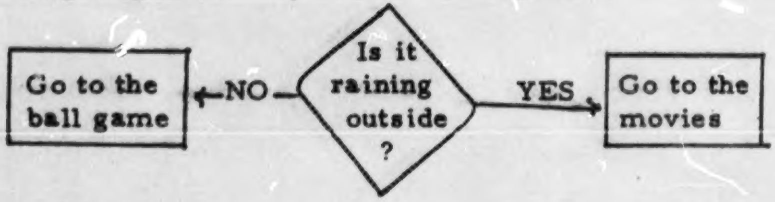
There is a useful word in the English language, anthropomorphize, which means: assign human characteristics to non-human things. Thus we have (often in children's fairy tales and nature stories) animals talking, flowers falling in love, or even a caterpillar smoking a pipe. Now anthropomorphization is being taken out of the world of literature and extended into the real world of modern science: some scientists speak seriously today of machines that think. We have seen them following instructions quickly and accurately. (For many of us those are not human traits.) But we have not yet let them perform their "most human" role: making decisions. We do so now.

What shall I do tonight? What TV channel should we watch? Shall I go to college? To answer these questions we make decisions: in these examples very complex decisions. As logicians analyze these questions they find that most can be reduced to one or more so-called binary questions, questions that have only two possible answers: yes or no.

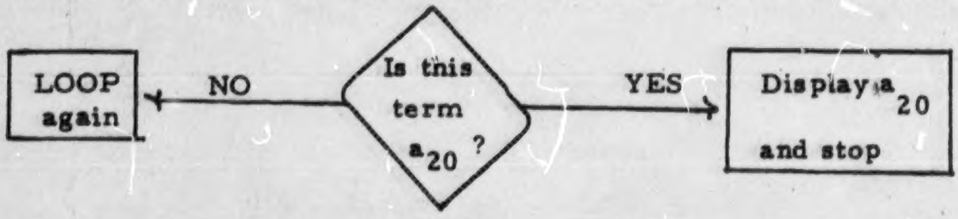
In numerical work the decisions may be further refined to questions like

Is $a = b$? Is $a > b$? Is $a \geq b$?

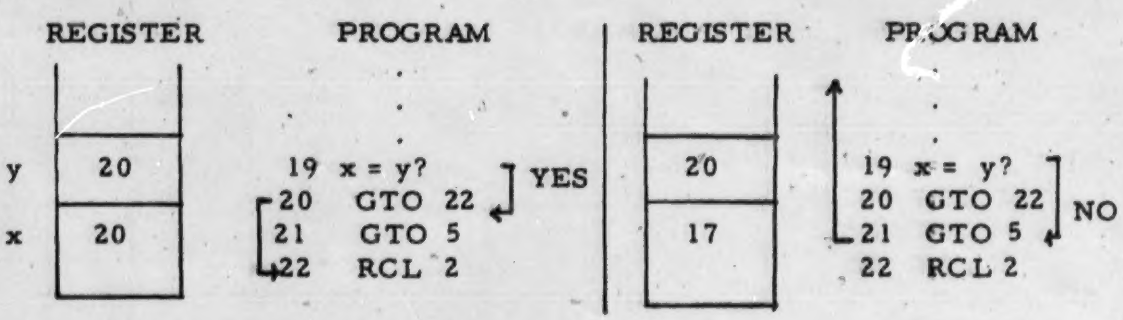
For specific values of a and b, these questions have yes-or-no answers. We can program the calculator to ask such questions and do different things depending on the answer. This closely mirrors the kind of human decision in the following diagram


Go to the ball game
NO
Is it raining outside ?
YES
Go to the movies

Suppose we wish to find the 20^{th} term of the sequence $\{2, 5, 8, \ldots\}$. In section 8.5 we programmed our calculator to count the number of the term so now we can examine that number to see if it is 20 as follows:


LOOP again
NO
Is this term a_{20} ?
YES
Display a_{20} and stop

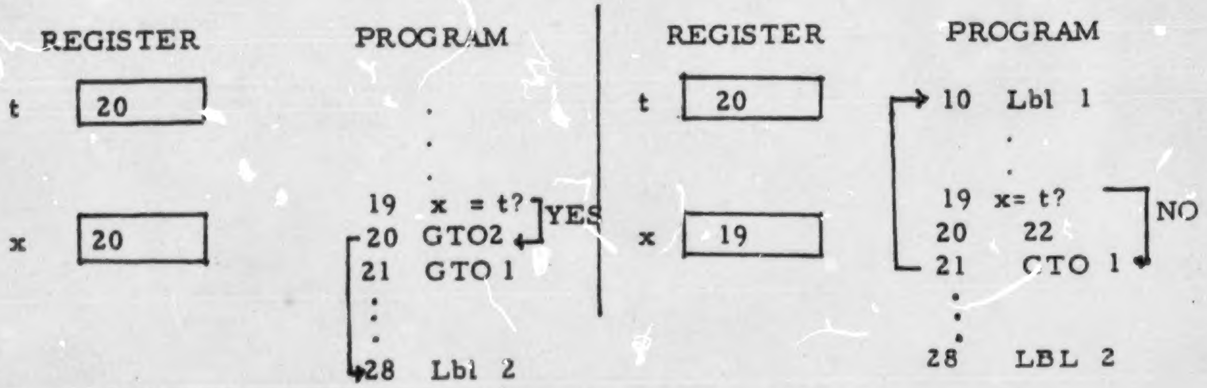
Different calculators have different decision making procedures. The HP-33E compares the x-register with the y-register. If the answer to the question asked is YES, the calculator continues to the next step; if the answer is NO, however, the calculator skips a step.


REGISTER
PROGRAM
y
20
x
20
YES
REGISTER
PROGRAM
20
17
NO

```
19  x = y?
20    GTO  22
21    GTO  5
22    RCL  2
```

```
19  x =  y?
20    GTO  22
21    GTO  5
22    RCL 2
```

The TI-57 does not have a y-register so a special store called a t-register is established. A number is placed in the t-register, R_t, by use of the $\boxed{x \gtrless t}$ key. Thus if we wish to place 20 in R_t, we merely key 20, $\boxed{x \lessgtr t}$. That number now becomes our comparison for the question, "Is $x = t$?" An example follows.


REGISTER
PROGRAM
t
20
x
20
19 x = t?
YES
20 GTO2
21 GTO 1
28 Lbl 2
REGISTER
PROGRAM
t
20
x
19
10 Lbl 1
19 x= t?
NO
20 22
21 GTO 1
28 LBL 2

The key to programming decisions is SKIP ON NO. This is a useful phrase to remember.

Now let's set up our program to find the 20th term of $\{2, 5, 8, \ldots\}$ So that we can use the program to find other steps we will store the 20 outside the program. Thus we will use

	HP-33E	TI-57
current term	R_0	R_0
number of current term	R_1	R_1
stop term number	R_2	R_t

Here is the program:

```
          HP-33E                      TI-57

           PRGM                        LRN

   01      STO  0              00    STO  0
                               01    0
   02      0                   02    STO  1
   03      STO  1              03    Lbl  1
   04      1                   04    1
   05      STO + 1             05    SUM  1
   06      RCL  2              06    RCL  1
   07      RCL  1              07    x = t?
   08      x = y?              08    GTO  2
   09      GTO  16             09    PAUSE  *
   10      PAUSE *             10    RCL  0  *
   11      RCL  0 *            11    PAUSE *
   12      PAUSE  *            12    3
   13      3                   13    SUM  0
   14      STO  + 0            14    GTO  1
   15      GTO  04             15    Lbl  2
   16      PAUSE               16    PAUSE
   17      RCL  0              17    RCL  0
   18      R/S                 18    R/S
           RUN RTN             19    RST
                                   LRN, RST
```

---

* These steps are included to display the loop results. For a faster program they may be omitted.

To run the program, key	HP-33E	TI-57
	20	20
	STO 2	$x \gtrless t$
	2	2
	R/S	R/S

Exercises 8.6

Two questions which have opposite answers to number comparisons are:

$$x = y? \qquad \text{and} \quad x \neq y?$$

Give the opposite question to each of the following:

1) $x > y$ 2) $x \geq y$ 3) $x < 0$

4) What would have happened to the programs of this section if you had used the decision key $\boxed{x > y?}$ or $\boxed{x \geq t?}$ instead of $\boxed{x = y?}$ or $\boxed{x = t?}$? (When calculations are involved it is often useful to use the inequality keys because of rounding problems.)

5) Develop a program for the n^{th} term of a geometric progression using the formula of section 8.4. Make it so that you would key in successively a, r, and n. Use it to find the 8^{th} term of $\{3, 1.5, .75, \ldots\}$.

6) Develop a program for the sum of n terms of an arithmetic progression, given a and ℓ and using a formula of section 8.3. Use it to find the sum of the odd numbers between 200 and 300. (How many are there?)

7) Develop a program like the one of this section to process the geometric sequence $\{3, 1.5, .75, \ldots\}$ calculating S_n for $n = 1, 2, 3, \ldots$.

Include a counter and a decision that will allow you to preset your program to stop at given n. Use $R_0 = a_n$, R_1 = n stopping (of current term) R_2 or R_t = term number, $R_3 = S_n$.

8) Modify your program in exercise 7 so that it will receive any rate r for a geometric progression. Use $R_0 = a_n$, R_1 = n (of current term), R_2 or R_t = stopping term number, $R_3 = S_n$ and $R_4 = r$.

9) Scientists involved in the field of Artificial Intelligence build many yes - no decisions like those of this section into very complex machines that guide airplanes, operate extensive sections of factory assembly lines, and analyze political situations. What are some human activities that you think computers can never achieve?

Chapter 8 TEST

Find the indicated term of each of the following sequences:

1) {10, 7, 4, 1, ...} 15th

2) {50, 52.50, 55, ...} 9th

3) {1024, 512, 256, ...} 12th

4) {3, 2.4, 1.92, ...} 6th

Find the sum of each sequence

5) {800, 750, 700 ...} 21 terms

6) {8, 4, 2, 1, ...} 10 terms

7) $\{5 - 3\cdot 1,\ 5 - 3\cdot 2,\ 5 - 3\cdot 3,\ 5 - 3\cdot 4,\ \ldots\}$ 20 terms

8) $a_1 = 81 \cdot {}^{2}/_{3}$

$a_2 = 81 \cdot {}^{4}/_{9}$

$a_3 = 81 \cdot {}^{8}/_{27}$

$S_7 = ?$

9) a) After working 30 days you are entitled to a single payment of a million dollars or 1¢ the first day, 2¢ the second, 4¢ the third and so on, doubling the amount you are paid on each succeeding day. Determine the better offer by careful calculation and indicate how much more you will receive.

10) Find the sum of the first two hundred natural numbers.

11) A piece of paper is folded in half 30 times. How high will the stack of paper be when finished? Assume the paper is .001 inches thick to start with.

12) A ball is dropped from the Empire State Building and it drops 16 ft. the first second, 48 ft. the second second, 80 ft. the third second, etc. Find, to the nearest tenth of a second, the time it will take to hit the ground.

Note: the Empire State Building is 1,472 ft. high.